Comparative Statics, Informativeness, and the Interval Dominance Order

John Quah and Bruno Strulovici

One-dimensional comparative statics

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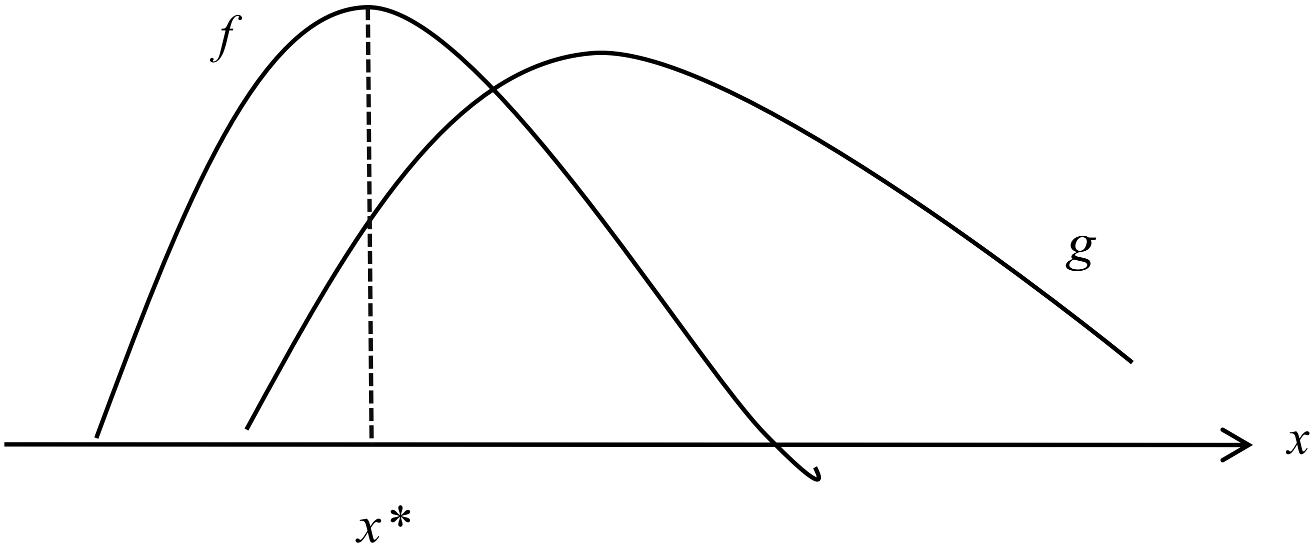

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Familiar argument:

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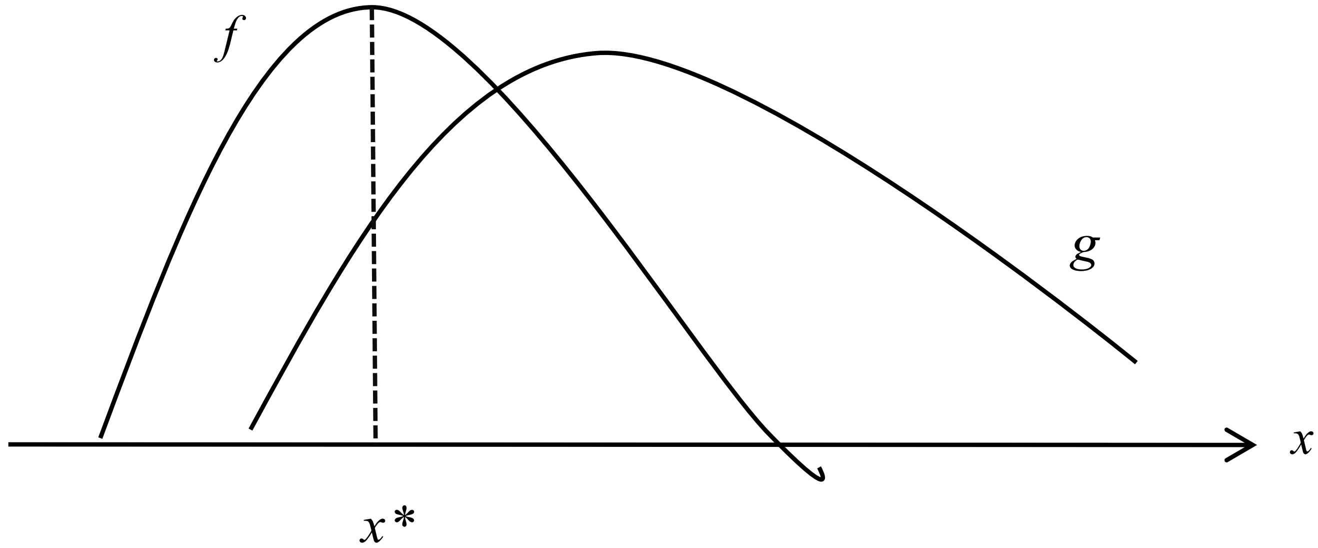

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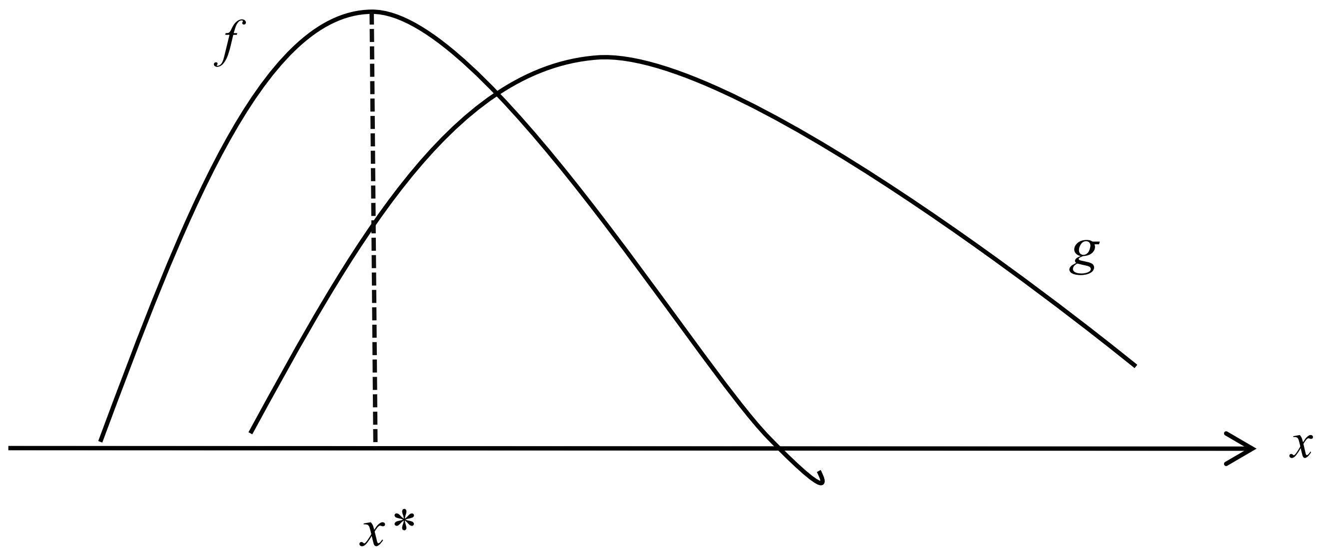

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Familiar argument: Since f is maximized at x^*, $f'(x^*) = 0$. Use this to show that $g'(x^*) > 0$.

This criterion is valid if f and g are quasiconcave functions, but not generally.

One-dimensional comparative statics

Consider two functions $f, g : X \to R$, with $X \subseteq R$.

g dominates f by the single crossing property ($g \succeq_{SC} f$) if for x^{**} and x^* with $x^{**} > x^*$, the following holds:

$$f(x^{**}) - f(x^*) \geq (>)\, 0 \;\Rightarrow\; g(x^{**}) - g(x^*) \geq (>)\, 0.$$

(Milgrom and Shannon (1994))

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$$f'(x) \geq (>)\, 0 \;\Rightarrow\; g'(x) \geq (>)\, 0.$$

The latter condition is useless when f and g are not quasiconcave.

One-dimensional comparative statics

When either $\text{argmax}_{x \in X} f(x)$ or $\text{argmax}_{x \in X} g(x)$ is non-singleton, how can the sets be ordered?

Definition: Let S' and S'' be subsets of R.
S'' dominates S' in the strong set order ($S'' \geq S'$) if for $x'' \in S''$ and $x' \in S'$,

$$\max\{x'', x'\} \in S'' \text{ and } \min\{x'', x'\} \in S'.$$

Example: $\{3, 5, 6, 7\} \not\geq \{1, 4, 6\}$

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Example: $\{3, 5, 6, 7\} \not\geq \{1, 4, 6\}$ but $\{3, 4, 5, 6, 7\} \geq \{1, 3, 4, 5, 6\}$.

Note: if $S'' = \{x''\}$ and $S' = \{x'\}$, then $x'' \geq x'$.

More generally,

$$\sup S'' \geq \sup S' \text{ and } \inf S'' \geq \inf S'.$$

The single crossing property

Theorem: Suppose f and g are real-valued functions defined on $X \subseteq R$ and $g \succeq_{SC} f$. Then $\text{argmax}_{x\in X} g(x) \geq \text{argmax}_{x\in X} f(x)$.

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Proof: Assume $g \succeq_{SC} f$ such that $x'' \in \text{argmax}_{x \in X} f(x)$ and $x' \in \text{argmax}_{x \in X} g(x)$ with $x'' > x'$.

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Proof: Assume $g \succeq_{SC} f$ such that $x'' \in \text{argmax}_{x \in X} f(x)$ and $x' \in \text{argmax}_{x \in X} g(x)$ with $x'' > x'$. Since $x'' \in \text{argmax}_{x \in X} f(x)$, we have $f(x'') \geq f(x')$. But $g \succeq_{SC} f$ implies that $g(x'') \geq g(x')$ so x'' is also in $\text{argmax}_{x \in X} g(x)$. And so on... QED

The single crossing property

Recall that $g \succeq_{SC} f$ if for x^{**} and x^* with $x^{**} > x^*$, the following holds:

$$f(x^{**}) - f(x^*) \geq (>)\, 0 \;\Rightarrow\; g(x^{**}) - g(x^*) \geq (>)\, 0. \tag{1}$$

A simple sufficient condition for $g \succeq_{SC} f$:

there is scalar $k > 0$ such that $g'(x) \geq kf'(x)$.

Clearly, this guarantees that

$$g(x^{**}) - g(x^*) \geq k\,[f(x^{**}) - f(x^*)]$$

and (1) follows.

The single crossing property

Simple Application: Consider the standard portfolio problem:

Agent V's utility function is $V(x) = \int v((w-x)r + xs)h(s)\,ds$,
– r is the payoff of the riskless asset; the risky asset pays s in state s, with the distribution of s given by the density function h
– x is the investment in the risky asset.

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Suppose Agent V is less risk averse than agent U; i.e., v has a smaller coefficient of risk aversion than u. Equivalently,

$\dfrac{v'(z)}{u'(z)}$ is increasing in z. This guarantees that

$$V'(x) \geq \frac{v'(wr)}{u'(wr)}\, U'(x).$$

So $V \succeq_{SC} U$ and comparative statics follows.
A bit more general than usual proof because neither v nor u need to be concave.

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Recall: $g \succeq_{SC} f$ if, for $x^{**} > x^*$, we have

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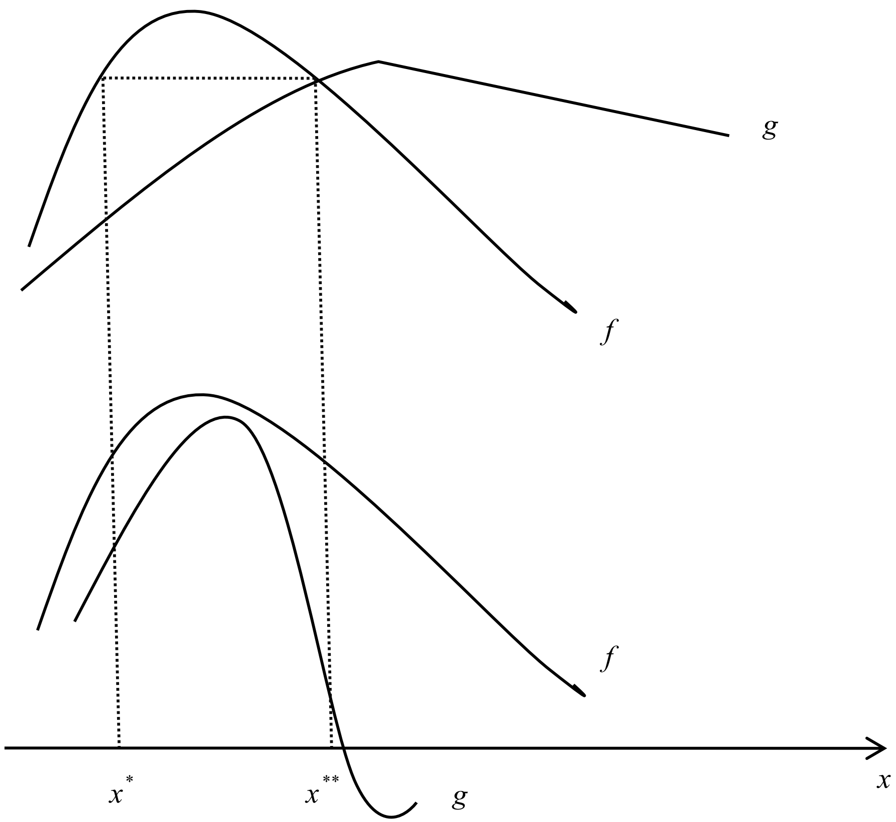

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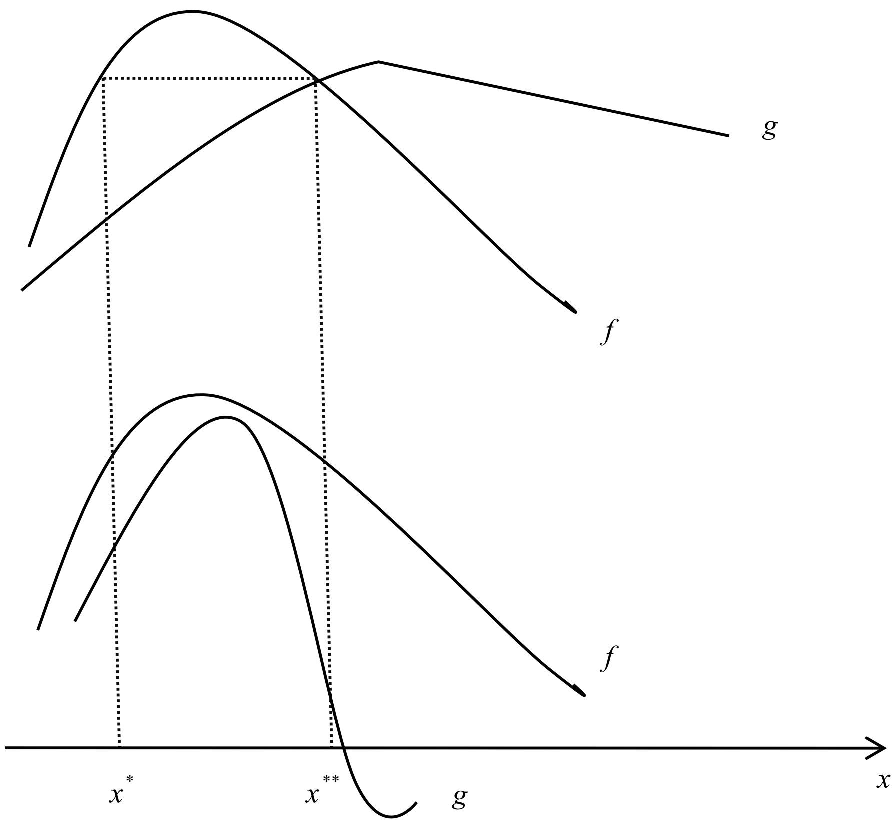

$g \succeq_{SC} f$ in top picture but not the lower.

The single crossing property

In both cases, the optimum has increased, but SCP captures just one of the two cases.

Motivation for the Interval Dominance Order:

to develop an ordering for functions that captures *both* situations.

The single crossing property

Let $X \subseteq R$ and $f, g : X \to R$. Recall: $g \succeq_{SC} f$ if, for any pair $x^{**} > x^*$, we have

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Alternatively, $g \succeq_{SC} f$ if for any pair $x^{**} > x^*$, we have

$$x^{**} \in \text{argmax}_{\{x^*, x^{**}\}} f(x) \quad \Rightarrow \quad x^{**} \in \text{argmax}_{\{x^*, x^{**}\}} g(x) \;\text{ and}$$
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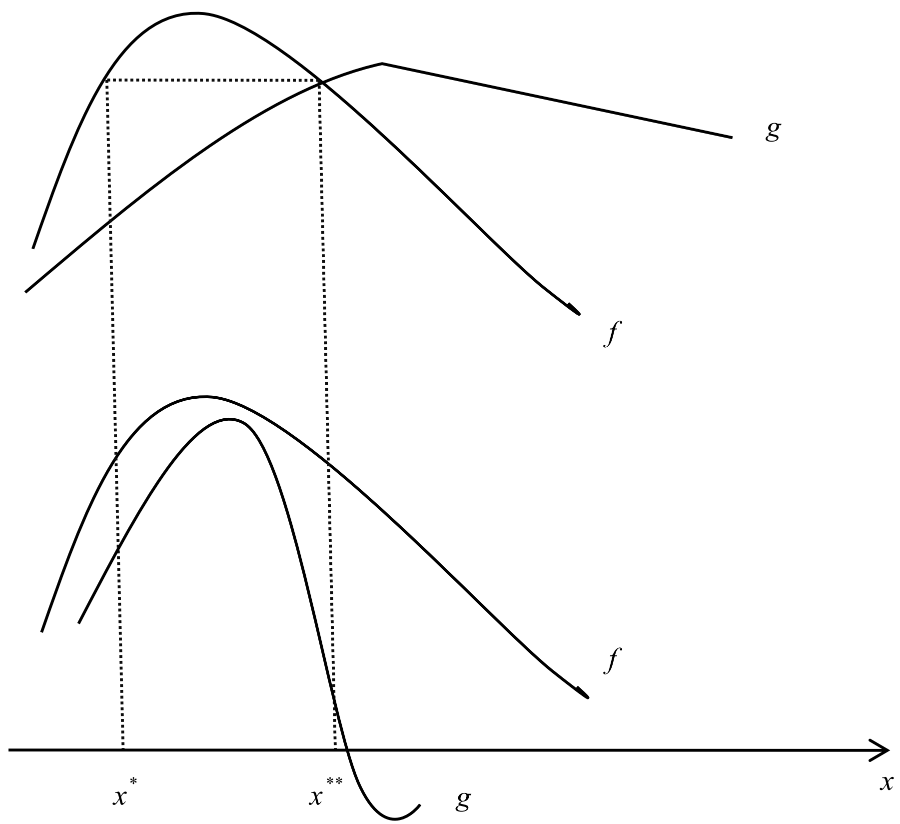

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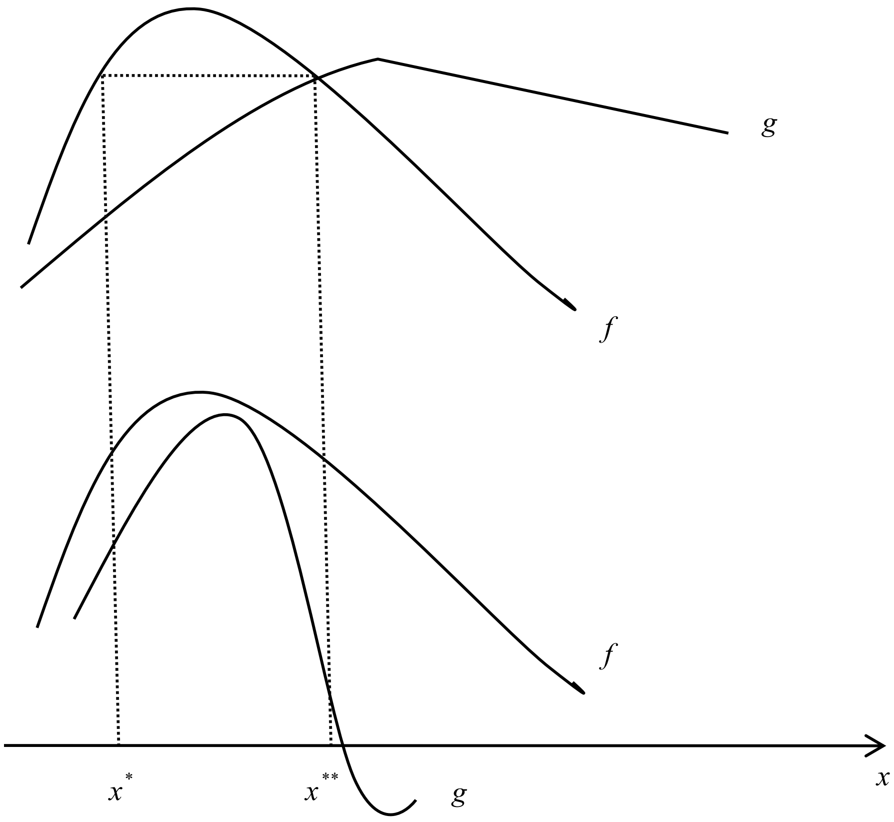

Notice that $(\star)$ *need not* be applied to x^{**} and x^*.

SCP and IDO

Both SCP and IDO are ordinal properties, i.e.,

if $g \succeq_{SC} f$, then $h \circ f \succeq_{SC} l \circ f$, where h and l are strictly increasing functions;

similarly,

if $g \succeq_{I} f$, then $h \circ f \succeq_{I} l \circ f$, where h and l are strictly increasing functions.

The interval dominance order

Theorem: Suppose that f and g are real-valued functions defined on $X \subset R$ and $g \succeq_I f$. Then $\text{argmax}_{x \in X} g(x) \geq \text{argmax}_{x \in X} f(x)$.

The interval dominance order

Theorem: Suppose that f and g are real-valued functions defined on $X \subset R$ and $g \succeq_I f$. Then argmax$_{x \in X} g(x) \geq$ argmax$_{x \in X} f(x)$.

Theorem can be made sharper:

A subset I of X is called an interval of X if for any x' and x'' in I and x in X such that $x' < x < x''$, we have x in I.

Note that X is always an interval of itself and
X need not be an interval of R.

Example: If $X = \{1, 2, 3, 5, 6\}$, then $\{1, 2, \}$ and $\{3, 5, 6\}$ are intervals of X but $\{3, 6\}$ is not an interval of X.

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Theorem: Suppose that f and g are real-valued functions defined on $X \subset R$. Then $g \succeq_I f$ if and only if

$\text{argmax}_{x \in I} g(x) \geq \text{argmax}_{x \in I} f(x)$ for any interval I of X.

Compare this with

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Theorem: Suppose that f and g are real-valued functions defined on $X \subset R$. Then $g \succeq_{SC} f$ if and only if
$\text{argmax}_{x \in S} g(x) \geq \text{argmax}_{x \in S} f(x)$ for any subset S of X.

The interval dominance order

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Sufficient condition for $g \succeq_{I} f$:

there is a nondecreasing positive function α such that

$$g'(x) \geq \alpha(x)f'(x).$$

The interval dominance order

The optimal stopping time problem: at each moment in time, agent gains profit of $\pi(t)$, which can be positive or negative. If agent decides to stop at time x, the present value of his accumulated profit is

$$V_\delta(x) = \int_0^x e^{-\delta t}\pi(t)dt$$

where $\delta > 0$ is the discount rate.

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where $\delta > 0$ is the discount rate.

How does optimal stopping time vary with discount rate?

Note that $V_\delta'(x) = e^{-\delta x}\pi(x)$. So

(i) there are lots of turning points;
(ii) turning points do not vary with the discount rate.

The interval dominance order

Proposition: Suppose

$$V_\delta(x) = \int_0^x e^{-\delta t}\pi(t)dt.$$

If $\delta > \bar{\delta} > 0$ then $\text{argmax}_{x\geq 0} V_{\bar{\delta}}(x) \geq \text{argmax}_{x\geq 0} V_\delta(x)$.

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Proof: We have

$$V'_{\bar{\delta}}(x) = e^{-\bar{\delta}x}\pi(x) = e^{(\delta-\bar{\delta})x}V'_\delta(x).$$

Note that the function $\alpha(x) = e^{(\delta-\bar{\delta})x}$ is positive and increasing.

So $V_{\bar{\delta}} \succeq_I V_\delta$. QED

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So $V_{\bar{\delta}} \succeq_I V_\delta$. QED

IDO with Uncertainty

Let $\{u(\cdot, s)\}_{s \in S}$ be an IDO family of functions, i.e.,

$$u(\cdot, s'') \succeq_I u(\cdot . s') \text{ whenever } s'' > s'.$$

Basic Example: for every s, $u(\cdot, s)$ is quasiconcave (in x) with $\operatorname{argmax} u(x, s'') \geq \operatorname{argmax} u(x, s')$ for $s'' > s'$.

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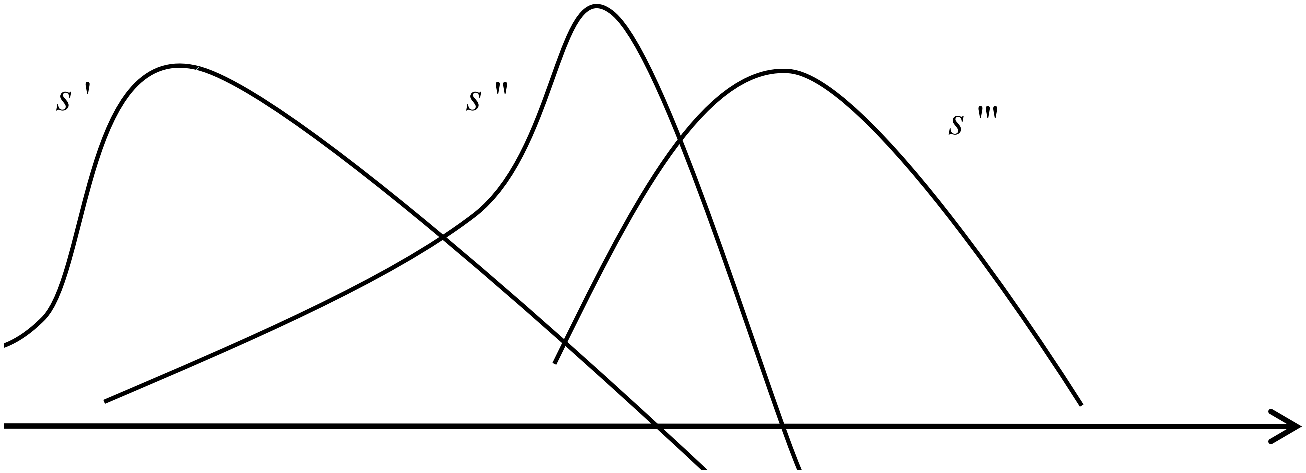

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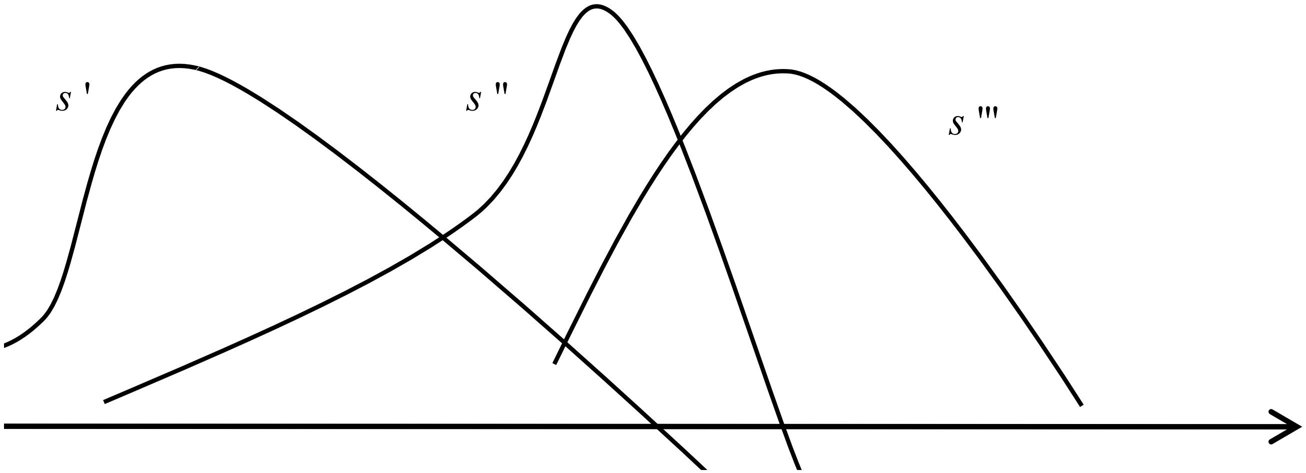
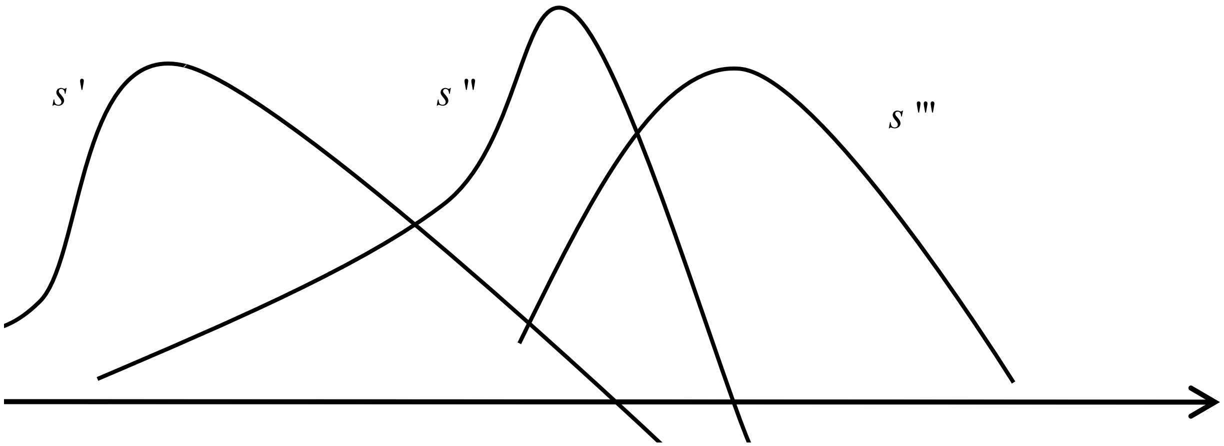


We call this the quasiconcave family with increasing peaks (QCIP). (Note: need not be an SCP family.)

IDO with Uncertainty

Each s denotes a state of the world...



At each state s there is an optimal action, which increases with s. Agent chooses x under uncertainty, i.e., before s is realized. He maximizes

$$U(x) = \int_{s \in S} u(x, s)\lambda(s) ds,$$

where $\lambda : S \to R$ is the density function.

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Theorem: Suppose $\{u(\cdot, s)\}_{s \in S}$ is an IDO family and let

$$U(x; \lambda) = \int_{s \in S} u(x, s)\lambda(s) ds.$$

Consider an MLR-shift in the density function from λ to $\bar{\lambda}$; i.e.,

$$\frac{\bar{\lambda}(s)}{\lambda(s)} \text{ is increasing in } s.$$

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Consider an MLR-shift in the density function from λ to $\bar{\lambda}$; i.e.,

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Then $U(\cdot, \bar{\lambda}) \succeq_I U(\cdot, \lambda)$. Consequently,

$$\text{argmax}_{x \in X} U(x; \bar{\lambda}) \geq \text{argmax}_{x \in X} U(x; \lambda).$$

Comparing Information Structures

Setting: Agent chooses action after observing a signal $z \in R$, but before realization of state.
Distribution over signals at a state s is $H(z|s)$ (with density $h(z|s)$).

The family $\{h(\cdot|s)\}_{s\in S}$ is the information structure H.

Assume that distributions are MLR-ordered, i.e., for $s'' > s'$

$$\frac{h(z|s'')}{h(z|s')} \text{ is increasing in } z.$$

Higher states make higher signals more likely.

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Suppose the agent is Bayesian, i.e., he has a unique prior P on the S. Given P, agent can work out the posterior distributions $\{\tilde{h}_P(\cdot|z)\}_{z\in Z}$.

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$$\phi(z) \in \text{argmax}_{x\in X} \int_{s\in S} u(x,s)\tilde{h}_P(s,z)ds$$

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In other words, higher signals lead to higher actions.

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Assume prior is P and consider information structure H. If ϕ_H is the optimal decision rule, then agent's ex ante utility is

$$\mathcal{V}(H,P) = \int_{z\in Z} \left[\int_{s\in S} u(\phi_H(z), s) d\tilde{h}_P(s|z) \right] d\nu_H$$

where ν_H is the marginal distribution of z.

We wish to compare H with another information structure $G = \{g(\cdot|s)\}_{s\in S}$. Suppose its optimal decision rule is ϕ_G, so

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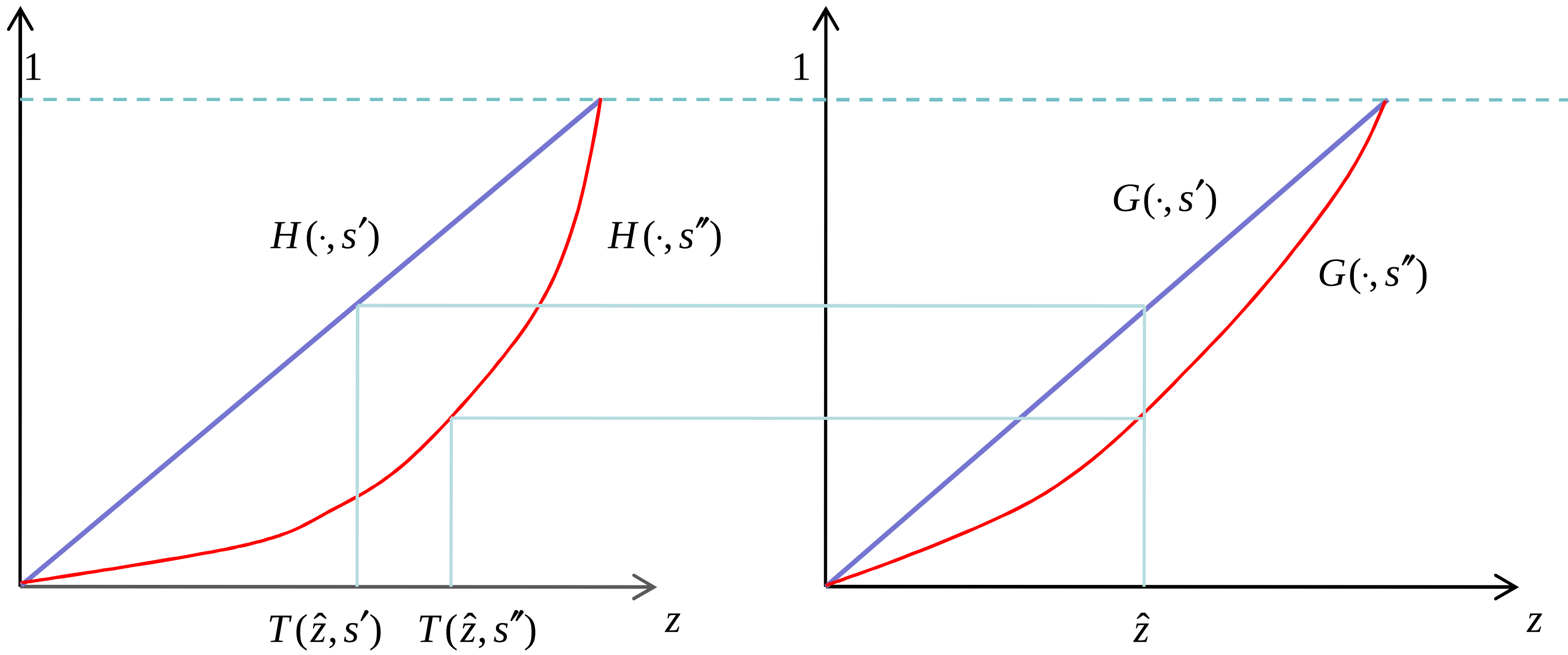
When is $\mathcal{V}(H,P) \geq \mathcal{V}(G,P)$ for all P?

Lehmann informativeness

Recall, $H(\cdot|s)$ is the distribution of signal z conditional on state s.
Similarly, $G(\cdot|s)$ is the distribution of signal z conditional on state s.
Define $T(z,s)$ by $H(T(z,s)|s) = G(z|s)$.

Definition: H is more informative than G (in the sense of Lehmann) if $T(z,\cdot)$ is increasing in s.



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Our extension: (ii) can be replaced with
(ii)$'$ $\{u(\cdot, s)\}_{s\in S}$ is an IDO family.

Statistical Decision Theory

Recall that if $H = \{h(\cdot|s)\}_{s \in S}$ is MLR-ordered, then the Bayesian agent has an *increasing* decision rule. In fact, increasing rules are appropriate even if agent uses some other decision criterion.

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The Bayesian (with prior P) chooses the rule that maximizes

$$\int_S \left[\int_Z u(\phi(z), s)dH(z|s)\right] dP(s).$$

The maxmin criterion chooses the rule that maximizes

$$\min_{s\in S} \int_Z u(\phi(z), s)dH(z|s).$$

Complete Class Theorem

Definition: A rule ϕ is better than ψ if

$$\left[\int_{z\in Z} u(\phi(z),s)h(z|s)\,dz\right]_{s\in S} \geq \left[\int_{z\in Z} u(\psi(z),s)h(z|s)\,dz\right]_{s\in S}$$

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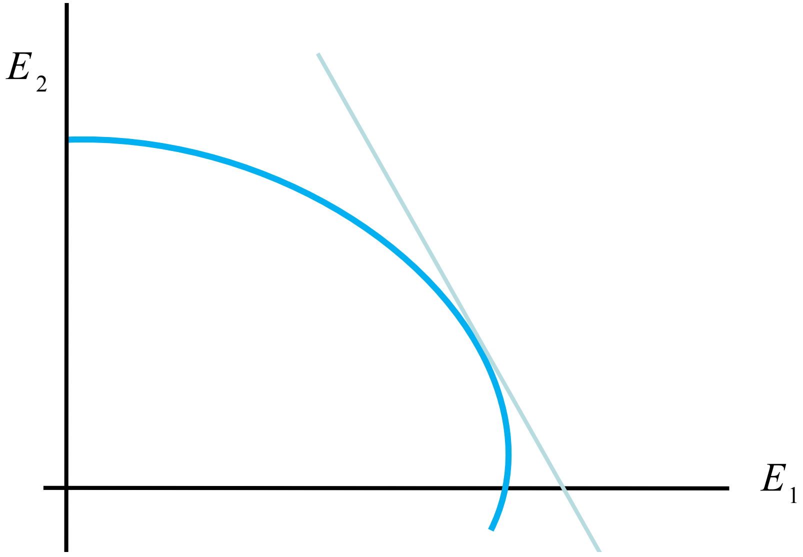
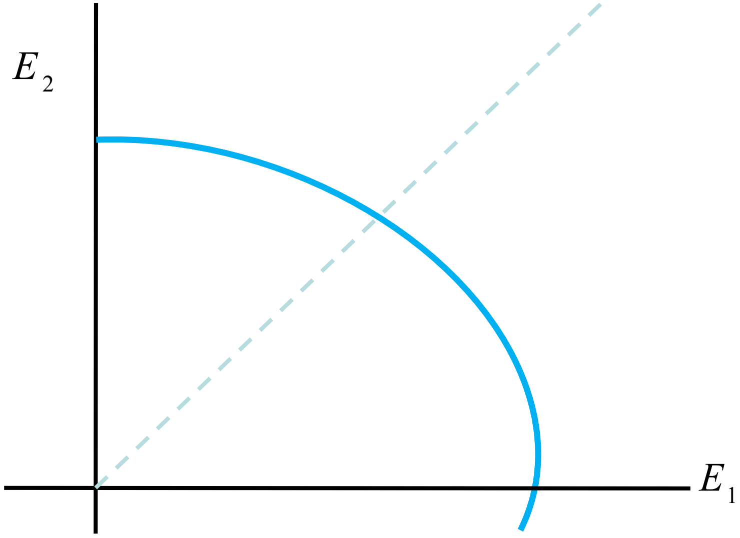
Geometrically.... when S is finite, we can think of $\left[\int_Z u(\phi(z),s)dH(z|s)\right]_{s\in S}$ as a point in $R^{|S|}$. Each rule is associated with a point in $R^{|S|}$.





$$[E_1, E_2] = \left[\int_{z\in Z} u(\phi(z),s_1)h(z|s_1)\,dz\,,\int_{z\in Z} u(\phi(z),s_2)h(z|s_2)\right]$$

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Our extension: QCIP family can replaced by IDO family.

Statistical Decision Theory

Application (adapted from Manski):

Medical Treatment A is the status quo with known recovery probability of $\bar{p}^A$.

Treatment B is the new treatment with unknown recovery probability of p^B, taking values in set S.

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Fact: The distribution of z given p^B is binomial and

$$\{h(\cdot|p^B)\}_{p^B \in S} \text{ is MLR-ordered.}$$

Statistical Decision Theory

Assume that cost of treating fraction x of the population with B (and the rest with A) is $C(x)$.

Normalize utility of cure at 1 and that of no cure at 0. Planner's utility if fraction x of the population receives B (and the rest A) is

$$u(x, p^B) = (1 - x)\,\bar{p}^A + x\,p^B - C(x). \tag{2}$$

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Conclusion (from extended Karlin-Rubin Theorem):
planner can confine herself to rules where x increases with z.

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Another application (to portfolio problem with multiple investors):

There is one safe asset and one risky asset.
There are N investors, with different risk attitudes and priors, who leave their funds with a *manager*.

Assume investors are rewarded the same share they put in, in all states of the world.

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Manager has information structure $H = \{h(\cdot|s)\}_{s \in S}$ (which we assume is MLR-ordered).
Manger has a decision rule which specifies the fraction of total funds, $\phi(z) \in [0, 1]$, to be invested in the risky asset after receiving signal z.

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Portfolio problem continued: Given a decision rule ϕ we can work out the ex ante expected utility of investors 1, 2, et al.

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Applying the extended versions of the Karlin-Rubin and Lehmann Theorems, we obtain:

1. Pareto optimal decision rules are increasing in signal.

2. If information structure improves in Lehmann's sense, then there is a new decision rule that leads to a Pareto improvement.