A revealed preference theory of monotone choice and strategic complementarity

Natalia Lazzati, John K.-H. Quah, and Koji Shirai

Background: Revealed Preference Analysis

Perhaps the most well known result in revealed preference analysis is Afriat's Theorem.

Afriat's theorem gives necessary and sufficient conditions under which a finite data set (consisting of prices and demand bundles) is consistent with the maximization of an increasing utility function.

Background: Revealed Preference Analysis

Perhaps the most well known result in revealed preference analysis is Afriat's Theorem.

Afriat's theorem gives necessary and sufficient conditions under which a finite data set (consisting of prices and demand bundles) is consistent with the maximization of an increasing utility function.

Revealed preference analysis in multi-agent contexts:

Brown and Matzkin (1996),
Sprumont (2000) and Ray and Zhou (2001),
Carvajal, Deb, Fenske, Quah (2013), amongst others.

The Bertrand oligopoly as supermodular game

Consider an oligopoly where firms compete in prices.

Firm i (for $i \in N$), with marginal cost $c_i > 0$, chooses the price of its product to maximize its profit π_i, where

$$\pi_i(p_i, p_{-i}; c_i) = (p_i - c_i) D_i(p_i, p_{-i}).$$

Reasonable assumptions on firm i's own price elasticity of demand, i.e., on

$$-\frac{p_i}{D_i} \frac{\partial D_i}{\partial p_i}(p_i, p_{-i}),$$

can guarantee that firm i's profit-maximizing price $\hat{p}_i$ is

(1) increasing with respect to the prices charged by other firms, p_{-i}.

Players' strategies in this game are strategic complements: a player's best response increases with the action of other players.

The Bertrand oligopoly as supermodular game

Consider an oligopoly where firms compete in prices.

Firm i (for $i \in N$), with marginal cost $c_i > 0$, chooses the price of its product to maximize its profit π_i, where

$$\pi_i(p_i, p_{-i}; c_i) = (p_i - c_i) D_i(p_i, p_{-i}).$$

Reasonable assumptions on firm i's own price elasticity of demand, i.e., on

$$-\frac{p_i}{D_i} \frac{\partial D_i}{\partial p_i}(p_i, p_{-i}),$$

can guarantee that firm i's profit-maximizing price $\hat{p}_i$ is

(1) increasing with respect to the prices charged by other firms, p_{-i}.

    Players' strategies in this game are strategic complements: a player's best response increases with the action of other players.

(2) Furthermore, $\hat{p}_i$ is increasing with respect to marginal cost, c_i.

The Bertrand oligopoly as supermodular game

Games with strategic complementarity are particularly well-behaved.

(1) There is always an equilibrium in *pure* strategies.

The Bertrand oligopoly as supermodular game

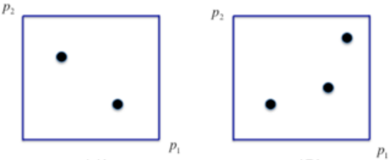
Games with strategic complementarity are particularly well-behaved.

(1) There is always an equilibrium in *pure* strategies.

(2) There is a *largest* Nash equilibrium, i.e., there is a Nash equilibrium $(\bar{p}_i)_{i\in\mathcal{I}}$ such that any other Nash equilibrium $(p_i')_{i\in\mathcal{I}}$ obeys

$$(\bar{p}_i)_{i\in\mathcal{I}} \geqslant (p_i')_{i\in\mathcal{I}}.$$

Similarly, there is a smallest Nash equilibrium.



(A) (B)

The set of equilibria cannot look like Figure (A).

Games with strategic complementarity

Let $N = \{1, 2, ..., n\}$ be the set of agents.

Let $X_i \subset \mathbb{R}$ be the set of all conceivable actions of agent i.

The feasible action set of agent i is $A_i \subset X_i$.

We assume that A_i is an interval of X_i, i.e.,

$$A_i = [\underline{a}_i, \bar{a}_i] = \{x_i \in X_i : \underline{a}_i \leqslant x_i \leqslant \bar{a}_i\}.$$

Games with strategic complementarity

Let $N = \{1, 2, ..., n\}$ be the set of agents.

Let $X_i \subset \mathbb{R}$ be the set of all conceivable actions of agent i.

The feasible action set of agent i is $A_i \subset X_i$.

We assume that A_i is an interval of X_i, i.e.,

$$A_i = [\underline{a}_i, \bar{a}_i] = \{x_i \in X_i : \underline{a}_i \leqslant x_i \leqslant \bar{a}_i\}.$$

There is an exogenous variable $y_i \in Y_i \subset \mathbb{R}$ affecting i's payoff.

Let $\Xi_i = X_{-i} \times Y_i$, where $X_{-i} := \times_{j \neq i} X_j$. Ξ_i is a poset if we endow it with the Euclidean order.

We denote a typical element of Ξ_i by $\xi_i = (a_{-i}; y_i)$.

A binary relation $\succsim_i$ on $X_i \times \Xi_i$ is a preference of agent i if, for every fixed $\xi_i \in \Xi_i$, $\succsim_i$ is a complete, reflexive and transitive relation on X_i.

Games with strategic complementarity

The preference $\succsim_i$ is said to be regular if, for every interval A_i and ξ_i, the best response set

$$\mathrm{BR}_i(\xi_i; A_i) = \{a_i \in A_i | (a_i, \xi_i) \succsim_i (x_i, \xi_i) \text{ for every } x_i \in A_i\} \neq \varnothing.$$

Given a profile of regular preferences $\{\succsim_i\}_{i \in N}$, a joint feasible action set $A = \times_{i \in N} A_i$, and a profile of exogenous parameters $y \in Y = \times_{i \in N} Y_i$, we can define a game

$$\mathcal{G}(A; y) = [N, (A_i)_{i \in N}, (\succsim_i)_{i \in N}; (y_i)_{i \in N}].$$

Letting A and y vary, we obtain the family of games

$$\mathbb{G} = \{\mathcal{G}(A, y)\}_{(A; y) \in \mathcal{A} \times Y}.$$

We say that $\mathbb{G}$ exhibits strategic complementarity if, for every interval A_i, $\mathrm{BR}_i(\xi_i; A_i)$ is increasing in ξ_i in the following sense: for every $\xi_i'' > \xi_i'$,

$$a_i'' \in \mathrm{BR}_i(\xi_i''; A_i) \text{ and } a_i' \in \mathrm{BR}_i(\xi_i'; A_i) \Longrightarrow a_i'' \geqslant a_i'.$$

Games with strategic complementarity

Games that exhibit strategic complementarity have very nice properties. (See Vives (1990), Milgrom and Roberts (1990), Zhou (1994).)

(1) Every game $\mathcal{G}(A, y)$ in $\mathbb{G}$ has a Nash equilibrium in pure strategies.

(2) Every game $\mathcal{G}(A, y)$ has a largest and a smallest Nash equilibrium.

(3) The largest (and smallest) Nash equilibrium of $\mathcal{G}(A, y)$ is increasing with respect to the exogenous parameters y.

Single Crossing Differences

Definition: The preference $\succsim_i$ on $X_i \times \Xi_i$ obeys strict single crossing differences if, for every $x_i'' > x_i'$ and $\xi_i'' > \xi_i'$,

$$(x_i'', \xi_i') \succsim_i (x_i', \xi_i') \Longrightarrow (x_i'', \xi_i'') \succ_i (x_i', \xi_i'').$$

Example: Firm i's profit function

$$\pi_i(p_i, p_{-i}; c_i) = (p_i - c_i) D_i(p_i, p_{-i})$$

obeys SCD, i.e., for $p_i'' > p_i'$ and $p_{-i}'' > p_{-i}'$

$$\pi_i(p_i'', p_{-i}') \geqslant \pi_i(p_i', p_{-i}') \Longrightarrow \pi_i(p_i'', p_{-i}'') > \pi_i(p_i', p_{-i}'')$$

if the own price elasticity of demand,

$$-\frac{p_i}{D_i}\frac{\partial D_i}{\partial p_i}(p_i, p_{-i}),$$

is strictly decreasing in p_{-i}.

Single Crossing Differences

Recall: by definition, the family of games

$$\mathbb{G} = \{\mathcal{G}(A, y)\}_{(A;y)\in\mathcal{A}\times Y}.$$

exhibits strategic complementarity if, for every interval A_i, $\mathrm{BR}_i(\xi_i; A_i)$ is increasing in $\xi_i = (a_{-i}, y_i)$, i.e., for every $\xi_i'' > \xi_i'$,

$$a_i'' \in \mathrm{BR}_i(\xi_i''; A_i) \text{ and } a_i' \in \mathrm{BR}_i(\xi_i'; A_i) \Longrightarrow a_i'' \geqslant a_i'.$$

Theorem 0: The family of games

$$\mathbb{G} = \{\mathcal{G}(A, y)\}_{(A;y)\in\mathcal{A}\times Y}$$

exhibits strategic complementarity if and only if for every $i \in N$, the preference relation $\succsim_i$ is regular and has strict single crossing differences.

Revealed Complementarity

Let $\mathbb{G} = \{\mathcal{G}(A, y)\}_{(A;y)\in\mathcal{A}\times Y}$ be a collection of games.

An observer has a *finite* set of observations drawn from this collection.

Each observation consists of agents' action profiles, feasible interval action sets, and exogenous parameters, i.e.,

$$(a^t, A^t; y^t),$$

where $a^t \in A^t$ is the observed action profile at the treatment (A^t, y^t) in $\mathcal{A} \times Y$. We denote the set of observations by

$$\mathcal{O} = \{(a^t, A^t; y^t)\}_{t\in\mathcal{T}}.$$

Definition: $\mathcal{O}$ is consistent with strategic complementarity if there exists a profile of regular preferences $\{\succsim_i\}_{i\in N}$ obeying single crossing differences such that each observation constitutes a Nash equilibrium, i.e.,

$$\text{for every } t \in \mathcal{T},\ (a^t_i, a^t_{-i}, y^t_i) \succsim_i (x_i, a^t_{-i}, y^t_i) \text{ for every } x_i \in A^t_i.$$

Revealed Complementarity

In other words, $\mathcal{O}$ is consistent with SC if and only if

for every agent i, there is a regular preference $\succsim_i$ obeying single crossing differences such that,

for all $t \in \mathcal{T}$, $(a_i^t, \xi_i^t) \succsim_i (x_i, \xi_i^t)$ for every $x_i \in A_i^t$ (where $\xi_i^t = (a_{-i}^t, y_i^t)$).

Revealed Complementarity

In other words, $\mathcal{O}$ is consistent with SC if and only if

for every agent i, there is a regular preference $\succsim_i$ obeying single crossing differences such that,

for all $t \in \mathcal{T}$, $(a_i^t, \xi_i^t) \succsim_i (x_i, \xi_i^t)$ for every $x_i \in A_i^t$ (where $\xi_i^t = (a_{-i}^t, y_i^t)$).

The direct revealed preference relation $\succsim_i^R$ is defined in the following way: $(x_i'', \xi_i) \succsim_i^R (x_i', \xi_i)$ if $(x_i'', \xi_i) = (a_i^t, \xi_i^t)$ and $x_i' \in A_i^t$ for some $t \in \mathcal{T}$.

Motivation of the terminology is clear: if agent i is maximizing some preference $\succsim_i$, then

$$(x_i'', \xi_i) \succsim_i^R (x_i', \xi_i) \Longrightarrow (x_i'', \xi_i) \succsim_i (x_i', \xi_i)$$

Revealed Complementarity

The direct revealed preference relation $\succsim_i^R$ is defined in the following way: $(x_i'', \xi_i) \succsim_i^R (x_i', \xi_i)$ if $(x_i'', \xi_i) = (a_i^t, \xi_i^t)$ and $x_i' \in A_i^t$ for some $t \in \mathcal{T}$.

The indirect revealed preference relation $\succsim_i^{RT}$ is the transitive closure of $\succsim_i^R$, i.e.,

$(x_i'', \xi_i) \succsim_i^{RT} (x_i', \xi_i)$ if there exists $z_i^1, z_i^2, ..., z_i^k$ in X_i such that

$$(x_i'', \xi_i) \succsim_i^R (z_i^1, \xi_i) \succsim_i^R (z_i^2, \xi_i) \succsim_i^R ... \succsim_i^R (z_i^k, \xi_i) \succsim_i^R (x_i', \xi_i).$$

Again, if agent i is maximizing some preference $\succsim_i$, then

$$(x_i'', \xi_i) \succsim_i^{RT} (x_i', \xi_i) \Longrightarrow (x_i'', \xi_i) \succsim_i (x_i', \xi_i)$$

Axiom of Revealed Complementarity

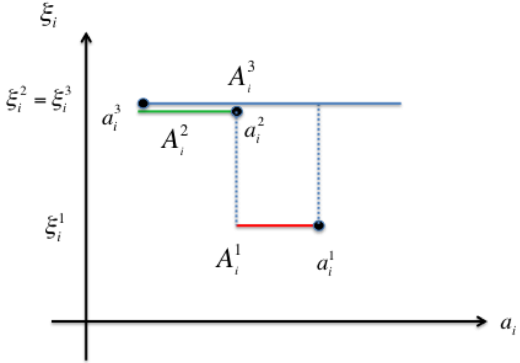
Definition: $\mathcal{O} = \{a^t, A^t; y^t\}_{t\in\mathcal{T}}$ obeys the axiom of revealed complementarity (ARC) when the following holds if, for every s, $t \in \mathcal{T}$,

$$\xi_i^t > \xi_i^s,\ a_i^t < a_i^s,\ \text{and } (a_i^s, \xi_i^s) \succsim_i^{RT} (a_i^t, \xi_i^s) \Longrightarrow (a_i^t, \xi_i^t) \not\succsim_i^{RT} (a_i^s, \xi_i^t).$$

Axiom of Revealed Complementarity

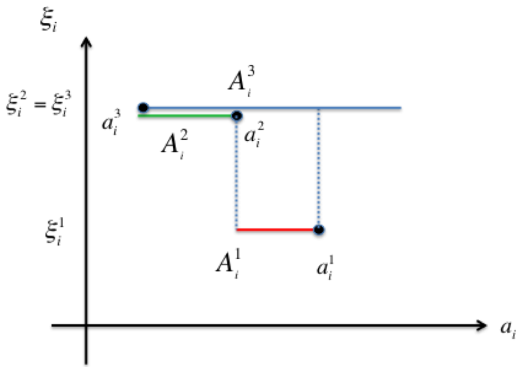
Definition: $\mathcal{O} = \{a^t, A^t; y^t\}_{t \in \mathcal{T}}$ obeys the axiom of revealed complementarity (ARC) when the following holds if, for every s, $t \in \mathcal{T}$,

$$\xi_i^t > \xi_i^s,\ a_i^t < a_i^s,\ \text{and } (a_i^s, \xi_i^s) \succsim_i^{RT} (a_i^t, \xi_i^s) \implies (a_i^t, \xi_i^t) \not\succsim_i^{RT} (a_i^s, \xi_i^t).$$



ARC violation: $(a_i^2, \xi_i^2) \succsim_i^{RT} (a_i^1, \xi_i^2)$ but $(a_i^1, \xi_i^1) \succsim_i^{R} (a_i^2, \xi_i^1)$

Axiom of Revealed Complementarity



ARC violation: $(a_i^2, \xi_i^2) \succsim_i^{RT} (a_i^1, \xi_i^2)$ but $(a_i^1, \xi_i^1) \succsim_i^{R} (a_i^2, \xi_i^1)$

This is not consistent with the existence of a preference $\succsim_i$ obeying single crossing differences.

$(a_i^1, \xi_i^1) \succsim_i^{R} (a_i^2, \xi_i^1)$ implies that $(a_i^1, \xi_i^1) \succsim_i (a_i^2, \xi_i^1)$. Single crossing differences then implies that $(a_i^1, \xi_i^2) \succ_i (a_i^2, \xi_i^2)$, so $(a_i^2, \xi_i^2) \not\succsim_i^{RT} (a_i^1, \xi_i^2)$.

Axiom of Revealed Complementarity

Definition: $\mathcal{O} = \{a^t, A^t; y^t\}_{t\in\mathcal{T}}$ obeys the axiom of revealed complementarity (ARC) if, for every s, $t \in \mathcal{T}$,

$$\xi_i^t > \xi_i^s,\ a_i^t < a_i^s,\ \text{and}\ (a_i^s, \xi_i^s) \succsim_i^{RT} (a_i^t, \xi_i^s) \Longrightarrow (a_i^t, \xi_i^t) \not\succ_i^{RT} (a_i^s, \xi_i^t).$$

Theorem 1:

$\mathcal{O} = \{a^t, A^t; y^t\}_{t\in\mathcal{T}}$ is consistent with SC if and only if it obeys ARC.

In the proof, we explicitly construct a preference $\succsim_i$ for player i that is regular, obeys single crossing differences, and rationalizes i's actions, i.e.,

$$\text{for every } t \in \mathcal{T},\ (a_i^t, \xi_i^t) \succsim_i (x_i, \xi_i^t) \text{ for every } x_i \in A_i^t.$$

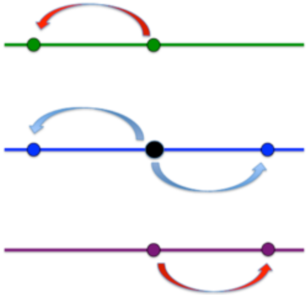
[Recall: $\xi_i^t = (a_{-i}^t, y^t)$.]

Sufficiency of ARC for SC-consistency

Definition: The single crossing extension of $\succsim_i^{RT}$ is the following binary relation $\succ_i^{RTS}$:

(i) for $x_i'' > x_i'$, $(x_i'', \xi_i) \succ_i^{RTS} (x_i', \xi_i)$ if there is $\xi_i' < \xi_i$ such that $(x_i'', \xi_i') \succsim_i^{RT} (x_i', \xi_i')$;

(ii) for $x_i'' < x_i'$, $(x_i'', \xi_i) \succ_i^{RTS} (x_i', \xi_i)$, if there is $\xi_i'' > \xi_i$ such that $(x_i'', \xi_i'') \succsim_i^{RT} (x_i', \xi_i'')$.



Sufficiency of ARC for SC-consistency

Definition: The single crossing extension of $\succsim_i^{RT}$ is the following binary relation $\succ_i^{RTS}$:

(i) for $x_i'' > x_i'$, $(x_i'', \xi_i) \succ_i^{RTS} (x_i', \xi_i)$ if there is $\xi_i' < \xi_i$ such that $(x_i'', \xi_i') \succsim_i^{RT} (x_i', \xi_i')$;

(ii) for $x_i'' < x_i'$, $(x_i'', \xi_i) \succ_i^{RTS} (x_i', \xi_i)$, if there is $\xi_i'' > \xi_i$ such that $(x_i'', \xi_i'') \succsim_i^{RT} (x_i', \xi_i'')$.

Given $\succ_i^{RTS}$, define $\succsim_i^{RTS} = \succsim_i^{RT} \cup \succ_i^{RTS}$.

Sufficiency of ARC for SC-consistency

Definition: The single crossing extension of $\succsim_i^{RT}$ is the following binary relation $\succ_i^{RTS}$:

(i) for $x_i'' > x_i'$, $(x_i'', \xi_i) \succ_i^{RTS} (x_i', \xi_i)$ if there is $\xi_i' < \xi_i$ such that $(x_i'', \xi_i') \succsim_i^{RT} (x_i', \xi_i')$;

(ii) for $x_i'' < x_i'$, $(x_i'', \xi_i) \succ_i^{RTS} (x_i', \xi_i)$, if there is $\xi_i'' > \xi_i$ such that $(x_i'', \xi_i'') \succsim_i^{RT} (x_i', \xi_i'')$.

Given $\succ_i^{RTS}$, define $\succsim_i^{RTS} = \succsim_i^{RT} \cup \succ_i^{RTS}$.

Clear that $\succsim_i^{RTS}$ also has single crossing differences:

if $x_i'' > x_i'$ and $\xi_i'' > \xi_i'$ *or* $x_i'' < x_i'$ and $\xi_i'' < \xi_i'$, then

$$(x_i'', \xi_i') \succsim_i^{RTS} (x_i', \xi_i') \Longrightarrow (x_i'', \xi_i'') \succ_i^{RTS} (x_i', \xi_i'').$$

Sufficiency of ARC for SC-consistency

Let $\succsim_i^{RTST}$ be the transitive closure of $\succsim_i^{RTS}$:

Sufficiency of ARC for SC-consistency

Let $\succsim_i^{RTST}$ be the transitive closure of $\succsim_i^{RTS}$:

$(x_i'', \xi_i) \succsim_i^{RTST} (x_i', \xi_i)$ if there exist $z_i^1, z_i^2, ..., z_i^k$ such that

$$(x_i'', \xi_i) \succsim_i^{RTS} (z_i^1, \xi_i) \succsim_i^{RTS} (z_i^2, \xi_i) \succsim_i^{RTS} ... \succsim_i^{RTS} (z_i^k, \xi_i) \succsim_i^{RTS} (x_i', \xi_i).$$

If we can find at least one strict relation $\succ_i^{RTS}$ is this sequence, then we define $(x_i'', \xi_i) \succ_i^{RTST} (x_i', \xi_i)$.

Sufficiency of ARC for SC-consistency

Let $\succsim_i^{RTST}$ be the transitive closure of $\succsim_i^{RTS}$:

$(x_i'', \xi_i) \succsim_i^{RTST} (x_i', \xi_i)$ if there exist $z_i^1, z_i^2, ..., z_i^k$ such that

$$(x_i'', \xi_i) \succsim_i^{RTS} (z_i^1, \xi_i) \succsim_i^{RTS} (z_i^2, \xi_i) \succsim_i^{RTS} ... \succsim_i^{RTS} (z_i^k, \xi_i) \succsim_i^{RTS} (x_i', \xi_i).$$

If we can find at least one strict relation $\succ_i^{RTS}$ is this sequence, then we define $(x_i'', \xi_i) \succ_i^{RTST} (x_i', \xi_i)$.

Proposition: Suppose the preference $\succsim_i$ obeys single crossing differences and rationalizes i's actions. Then $\succsim_i$ extends $\succsim_i^{RTST}$ and $\succ_i^{RTST}$, i.e.,

$$(x_i'', \xi_i) \succsim_i^{RTST} (\succ_i^{RTST}) (x_i', \xi_i) \Longrightarrow (x_i'', \xi_i) \succsim_i (\succ_i) (x_i', \xi_i)$$

Sufficiency of ARC for SC-consistency

Let $\succsim_i^{RTST}$ be the transitive closure of $\succsim_i^{RTS}$:

$(x_i'', \xi_i) \succsim_i^{RTST} (x_i', \xi_i)$ if there exist $z_i^1, z_i^2, ..., z_i^k$ such that

$$(x_i'', \xi_i) \succsim_i^{RTS} (z_i^1, \xi_i) \succsim_i^{RTS} (z_i^2, \xi_i) \succsim_i^{RTS} ... \succsim_i^{RTS} (z_i^k, \xi_i) \succsim_i^{RTS} (x_i', \xi_i).$$

If we can find at least one strict relation $\succ_i^{RTS}$ is this sequence, then we define $(x_i'', \xi_i) \succ_i^{RTST} (x_i', \xi_i)$.

Proposition: Suppose the preference $\succsim_i$ obeys single crossing differences and rationalizes i's actions. Then $\succsim_i$ extends $\succsim_i^{RTST}$ and $\succ_i^{RTST}$, i.e.,

$$(x_i'', \xi_i) \succsim_i^{RTST} (\succ_i^{RTST}) (x_i', \xi_i) \Longrightarrow (x_i'', \xi_i) \succsim_i (\succ_i) (x_i', \xi_i)$$

Note: In fact, every preference $\succsim_i$ with single crossing differences that rationalizes i's actions must also extend $\succsim_i^{RTSTS}$, $\succsim_i^{RTSTST}$, etc.

Sufficiency of ARC for SC-consistency

Let $\succsim_i^{RTST}$ be the transitive closure of $\succsim_i^{RTS}$:

$(x_i'', \xi_i) \succsim_i^{RTST} (x_i', \xi_i)$ if there exist $z_i^1, z_i^2, ..., z_i^k$ such that

$$(x_i'', \xi_i) \succsim_i^{RTS} (z_i^1, \xi_i) \succsim_i^{RTS} (z_i^2, \xi_i) \succsim_i^{RTS} ... \succsim_i^{RTS} (z_i^k, \xi_i) \succsim_i^{RTS} (x_i', \xi_i).$$

If we can find at least one strict relation $\succ_i^{RTS}$ is this sequence, then we define $(x_i'', \xi_i) \succ_i^{RTST} (x_i', \xi_i)$.

Proposition: Suppose the preference $\succsim_i$ obeys single crossing differences and rationalizes i's actions. Then $\succsim_i$ extends $\succsim_i^{RTST}$ and $\succ_i^{RTST}$, i.e.,

$$(x_i'', \xi_i) \succsim_i^{RTST} (\succ_i^{RTST}) (x_i', \xi_i) \Longrightarrow (x_i'', \xi_i) \succsim_i (\succ_i) (x_i', \xi_i)$$

Note: In fact, every preference $\succsim_i$ with single crossing differences that rationalizes i's actions must also extend $\succsim_i^{RTSTS}$, $\succsim_i^{RTSTST}$, etc.

However, there is no need to go beyond $\succsim_i^{RTST}$ because

$$\succsim_i^{RTST} = \succsim_i^{RTSTS} = \succsim_i^{RTSTST} = ...$$

Sufficiency of ARC for SC-consistency

Key conclusion: Every regular and preference $\succsim_i$ that rationalizes i's action *must* extend from $\succsim_i^{RTST}$ and $\succ_i^{RTST}$.

Sufficiency of ARC for SC-consistency

Key conclusion: Every regular and preference $\succsim_i$ that rationalizes i's action *must* extend from $\succsim_i^{RTST}$ and $\succ_i^{RTST}$.

The following is a regular preference obeying single crossing differences that rationalizes i's actions:

$$(x_i'', \xi_i) \succsim_i^* (x_i', \xi_i) \qquad \begin{array}{l} \text{if } (x_i'', \xi_i) \succsim_i^{RTST} (x_i', \xi_i) \\ \text{or } (x_i'', \xi_i) \parallel_i^{RTST} (x_i', \xi_i) \text{ and } x_i' \geqslant x_i''. \end{array}$$

Sufficiency of ARC for SC-consistency

Key conclusion: Every regular and preference $\succsim_i$ that rationalizes i's action *must* extend from $\succsim_i^{RTST}$ and $\succ_i^{RTST}$.

The following is a regular preference obeying single crossing differences that rationalizes i's actions:

$$(x_i'', \xi_i) \succsim_i^* (x_i', \xi_i) \qquad \begin{array}{l} \text{if } (x_i'', \xi_i) \succsim_i^{RTST} (x_i', \xi_i) \\ \text{or } (x_i'', \xi_i) \parallel_i^{RTST} (x_i', \xi_i) \text{ and } x_i' \geqslant x_i''. \end{array}$$

Taking stock...

We have shown how a data set can be tested for strategic complementarity.

Theorem 1: A dataset $\mathcal{O} = \{a^t, A^t; y^t\}_{t \in \mathcal{T}}$ is consistent with SC if and only if it obeys ARC.

Taking stock...

We have shown how a data set can be tested for strategic complementarity.

Theorem 1: A dataset $\mathcal{O} = \{a^t, A^t; y^t\}_{t \in \mathcal{T}}$ is consistent with SC if and only if it obeys ARC.

Follow-up question:

What can we say about the possible Nash equilibria in a new game with the same players but different strategy sets and different exogenous variables?

How can we compute the set of possible Nash equilibria?

Identification of Possible Nash Equilibria

Let $\mathcal{O} = (a^t, A^t; y^t)$ be a data set that obeys ARC.

Denote by $\mathcal{P}_i^*$ the set of preferences for player i that rationalize player i's actions and obey single crossing differences.

We denote $\times_{i=1}^N \mathcal{P}_i^*$ by $\mathcal{P}^*$.

Identification of Possible Nash Equilibria

Let $\mathcal{O} = (a^t, A^t; y^t)$ be a data set that obeys ARC.

Denote by $\mathcal{P}_i^*$ the set of preferences for player i that rationalize player i's actions and obey single crossing differences.

We denote $\times_{i=1}^N \mathcal{P}_i^*$ by $\mathcal{P}^*$.

Consider a new game $\mathcal{G}(A^0, y^0)$, where $A^0 = \times_{i=1}^N A_i^0$ and $y^0 = (y_i^0)_{i=1}^N$.

Define

$$\Gamma(A^0, y^0) = \{\bar{a} \in A^0 : \exists(\succsim_i)_{i \in N} \in \mathcal{P}^* \text{ s.t. } \bar{a} \in NE[\mathcal{G}((\succsim_i)_{i \in N}, A^0, y^0)]\}$$

This is the set of possible Nash equilibria of $\mathcal{G}(A^0, y^0)$ that is consistent with $\mathcal{O}$ and strategic complementarity.

Identification of Possible Nash Equilibria

Let $\mathcal{O} = (a^t, A^t; y^t)$ be a data set that obeys ARC.

Denote by $\mathcal{P}_i^*$ the set of preferences for player i that rationalize player i's actions and obey single crossing differences.

We denote $\times_{i=1}^N \mathcal{P}_i^*$ by $\mathcal{P}^*$.

Consider a new game $\mathcal{G}(A^0, y^0)$, where $A^0 = \times_{i=1}^N A_i^0$ and $y^0 = (y_i^0)_{i=1}^N$.

Define

$$\Gamma(A^0, y^0) = \{\bar{a} \in A^0 : \exists (\gtrsim_i)_{i \in N} \in \mathcal{P}^* \text{ s.t. } \bar{a} \in NE[\mathcal{G}((\gtrsim_i)_{i \in N}, A^0, y^0)]\}$$

This is the set of possible Nash equilibria of $\mathcal{G}(A^0, y^0)$ that is consistent with $\mathcal{O}$ and strategic complementarity.

Clearly, $\Gamma(A^0, y^0)$ is nonempty since $\mathcal{P}^*$ is nonempty.

How can we construct $\Gamma(A^0, y^0)$ from the data?

Identification of Possible Nash Equilibria

$\Gamma(A^0, y^0)$ is the set of fixed points of a correspondence.

For each player i and $\xi_i^0 = (a_i, y_i^0) \in A_{-i}^0 \times \{y_i^0\}$, we define

$$P_i(\xi_i^0) = \left\{a_i \in A_i^0 : \exists \succsim_i \in \mathcal{P}_i^* \text{ s.t. } (a_i, \xi_i^0) \succsim_i (\tilde{a}_i, \xi_i^0) \forall \tilde{a}_i \in A_i^0\right\}$$

This is the set of possible best responses of player i.

Identification of Possible Nash Equilibria

$\Gamma(A^0, y^0)$ is the set of fixed points of a correspondence.

For each player i and $\xi_i^0 = (a_i, y_i^0) \in A_{-i}^0 \times \{y_i^0\}$, we define

$$P_i(\xi_i^0) = \left\{a_i \in A_i^0 : \exists \succsim_i \in \mathcal{P}_i^* \text{ s.t. } (a_i, \xi_i^0) \succsim_i (\tilde{a}_i, \xi_i^0) \forall \tilde{a}_i \in A_i^0\right\}$$

This is the set of possible best responses of player i.

Crucially, this correspondence can be computed, since

$$P_i(\xi_i^0) = \{a_i \in A_i^0 : \nexists \hat{a}_i \in A_i^0 \text{ such that } (\hat{a}_i, \xi_i^0) \succ_i^{RTST} (a_i, \xi_i^0)\}.$$

Identification of Possible Nash Equilibria

$\Gamma(A^0, y^0)$ is the set of fixed points of a correspondence.

For each player i and $\xi_i^0 = (a_i, y_i^0) \in A_{-i}^0 \times \{y_i^0\}$, we define

$$P_i(\xi_i^0) = \left\{a_i \in A_i^0 : \exists \succsim_i \in \mathcal{P}_i^* \text{ s.t. } (a_i, \xi_i^0) \succsim_i (\tilde{a}_i, \xi_i^0) \forall \tilde{a}_i \in A_i^0\right\}$$

This is the set of possible best responses of player i.

Crucially, this correspondence can be computed, since

$$P_i(\xi_i^0) = \{a_i \in A_i^0 : \nexists \hat{a}_i \in A_i^0 \text{ such that } (\hat{a}_i, \xi_i^0) \succ_i^{RTST} (a_i, \xi_i^0)\}.$$

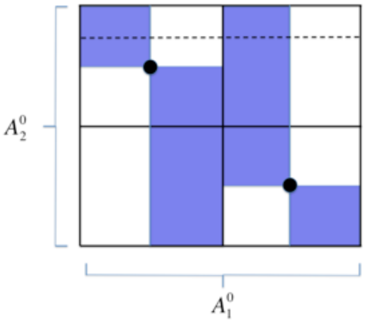
We define the possible response correspondence $P : A^0 \to A^0$ by

$$P(a) = (P_i(a_{-i}, y_i^0))_{i=1}^N,$$

where $a = (a_1, a_2, ..., a_N)$ and $a_{-i} = (a_i)_{j \neq i}$.

Then $\Gamma(A^0, y^0) =$ Fixed points of P.

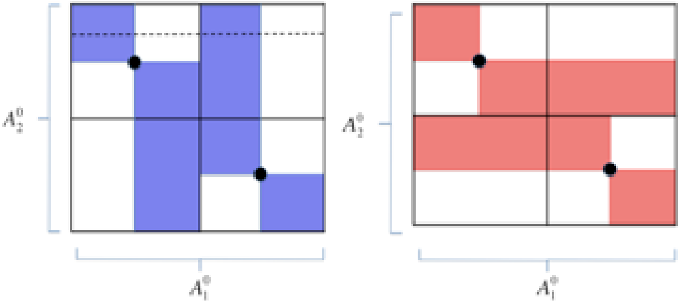
Identification of Possible Nash Equilibria



In the game with strategy set $A_1^0 \times A_2^0$, only the unshaded area can be NE.

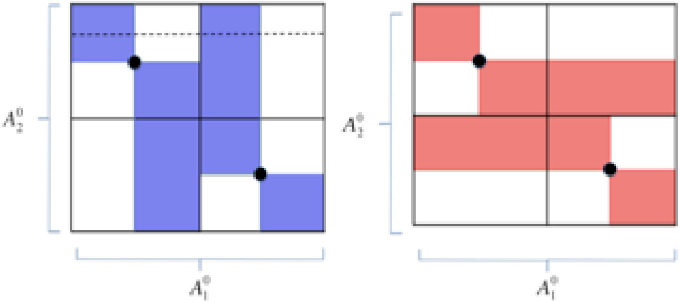
In the shaded area, player 1 cannot be playing his best response.

Identification of Possible Nash Equilibria

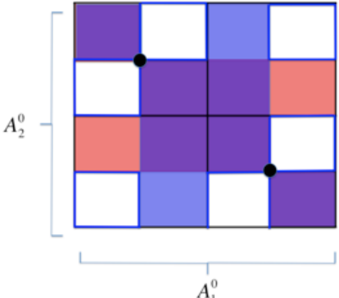


The common unshaded areas represent $\Gamma(A^0)$.

Identification of Possible Nash Equilibria



The common unshaded areas represent $\Gamma(A^0)$.



Structure of Possible Nash Equilibria

Theorem 2: Suppose $\mathcal{O} = \{(a^t, A^t; y^t)\}_{t\in\mathcal{T}}$ obeys ARC.

Then $\Gamma(A^0, y^0)$ has the following properties:

(1) $\sup \Gamma(A^0, y^0)$ and $\inf \Gamma(A^0, y^0)$ are in the closure of $\Gamma(A^0, y^0)$.

Structure of Possible Nash Equilibria

Theorem 2: Suppose $\mathcal{O} = \{(a^t, A^t; y^t)\}_{t \in \mathcal{T}}$ obeys ARC.

Then $\Gamma(A^0, y^0)$ has the following properties:

(1) $\sup \Gamma(A^0, y^0)$ and $\inf \Gamma(A^0, y^0)$ are in the closure of $\Gamma(A^0, y^0)$.

In particular, when X_i is finite for all i, then $\sup \Gamma(A^0, y^0) \in \Gamma(A^0, y^0)$ and $\inf \Gamma(A^0, y^0) \in \Gamma(A^0, y^0)$.

Structure of Possible Nash Equilibria

Theorem 2: Suppose $\mathcal{O} = \{(a^t, A^t; y^t)\}_{t\in\mathcal{T}}$ obeys ARC.

Then $\Gamma(A^0, y^0)$ has the following properties:

(1) $\sup\Gamma(A^0, y^0)$ and $\inf\Gamma(A^0, y^0)$ are in the closure of $\Gamma(A^0, y^0)$.

In particular, when X_i is finite for all i, then
$\sup\Gamma(A^0, y^0) \in \Gamma(A^0, y^0)$ and $\inf\Gamma(A^0, y^0) \in \Gamma(A^0, y^0)$.

(2) $\sup\Gamma(A^0, y^0)$ and $\inf\Gamma(A^0, y^0)$ are increasing in y^0.

Structure of Possible Nash Equilibria

Theorem 2: Suppose $\mathcal{O} = \{(a^t, A^t; y^t)\}_{t \in \mathcal{T}}$ obeys ARC.

Then $\Gamma(A^0, y^0)$ has the following properties:

(1) $\sup \Gamma(A^0, y^0)$ and $\inf \Gamma(A^0, y^0)$ are in the closure of $\Gamma(A^0, y^0)$.

In particular, when X_i is finite for all i, then
$\sup \Gamma(A^0, y^0) \in \Gamma(A^0, y^0)$ and $\inf \Gamma(A^0, y^0) \in \Gamma(A^0, y^0)$.

(2) $\sup \Gamma(A^0, y^0)$ and $\inf \Gamma(A^0, y^0)$ are increasing in y^0.

Both properties echo the structure of the set Γ of *actual* Nash equilibria of a game with strategic complements: in that case,

(1) $\sup \Gamma \in \Gamma$ and $\inf \Gamma \in \Gamma$ and

(2) $\max \Gamma$ and $\min \Gamma$ both increase with y^0.

Cross sectional data sets

So far in our analysis, we assume we have access to panel data, where the same agent or group is observed under different treatments (at different times).

Our analysis can also be applied to cross-sectional data, where we observe populations of groups of interacting agents, with heterogeneous preferences, subject to different treatments.

Cross sectional data sets

So far in our analysis, we assume we have access to panel data, where the same agent or group is observed under different treatments (at different times).

Our analysis can also be applied to cross-sectional data, where we observe populations of groups of interacting agents, with heterogeneous preferences, subject to different treatments.

We can test the hypothesis that these groups are playing games of strategic complementarity, under the following assumption:

the distribution of preferences is the same across populations subject to different treatments.

Cross sectional data sets

So far in our analysis, we assume we have access to panel data, where the same agent or group is observed under different treatments (at different times).

Our analysis can also be applied to cross-sectional data, where we observe populations of groups of interacting agents, with heterogeneous preferences, subject to different treatments.

We can test the hypothesis that these groups are playing games of strategic complementarity, under the following assumption:

the distribution of preferences is the same across populations subject to different treatments.

This treatment is analogous to McFadden and Richter's extension to cross sectional data of revealed preference tests of utility-maximization.

Cross sectional data sets

We observe a large population of *groups*, with n players in each group.

Example: data on a large population of husband-wife pairs.

Members of a group play a game

$$\mathcal{G}(A; y) = [N, (A_i)_{i \in N}, (\succsim_i)_{i \in N}; (y_i)_{i \in N}]$$

(Assume that all players have finite strategy sets.)

At observation t, the game being played is $\mathcal{G}(A^t, y^t)$.

Groups in the population choose different action combinations.

We observe the distribution μ^t over the set of action profiles, $A^t = \times_{i=1}^N A_i^t$.

Thus we have an (idealized) data set $\mathcal{O} = \{(\mu^t, A^t, y^t)\}_{t \in \mathcal{T}}$.

Cross sectional data sets

We observe a large population of *groups*, with n players in each group.

Example: data on a large population of husband-wife pairs.

Members of a group play a game

$$\mathcal{G}(A; y) = [N, (A_i)_{i\in N}, (\succsim_i)_{i\in N}; (y_i)_{i\in N}]$$

(Assume that all players have finite strategy sets.)

At observation t, the game being played is $\mathcal{G}(A^t, y^t)$.

Groups in the population choose different action combinations.

We observe the distribution μ^t over the set of action profiles, $A^t = \times_{i=1}^N A_i^t$.

Thus we have an (idealized) data set $\mathcal{O} = \{(\mu^t, A^t, y^t)\}_{t\in\mathcal{T}}$.

We allow for $(A^t, y^t) = (A^s, y^s)$ and $\mu^t \neq \mu^s$. In other words, selection may differ between t and s, though treatment is the same.

Cross sectional data sets

Let $\mathcal{O} = \{(\mu^t, A^t, y^t)\}_{t \in \mathcal{T}}$ be a (cross-sectional) data set.

We denote the set of treatments by $E = \{(A^t, y^t)\}_{t \in \mathcal{T}}$.

We call

$$\hat{a} = (\hat{a}^1, \hat{a}^2, ..., \hat{a}^T)$$

an SC-consistent path on E if the (panel) data set $\{(\hat{a}^t, A^t, y^t)\}_{t=1}^T$ is SC-consistent.

By Theorem 1:

$\hat{a}$ is an SC-consistent path on E if and only if it is an ARC-consistent path on E, i.e., $\{(\hat{a}^t, A^t, y^t)\}_{t=1}^T$ obeys ARC.

In general, there are multiple SC-consistent paths on E.

We denote this set by A^*.

Cross sectional data sets

Definition: A dataset $\mathcal{O} = \{\mu^t, A^t, y^t\}_{t\in\mathcal{T}}$ is consistent with strategic complementarity if there exists a probability distribution Q on A^*, the set of SC-consistent paths in the set of environments $E = \{(A^t, y^t)\}_{t\in\mathcal{T}}$, such that, for each $a \in A^t$ and $t \in \mathcal{T}$,

$$\mu^t(a) = \sum_{\hat{a}\in A*} Q(\hat{a})\mathbf{1}(\hat{a}^t = a).$$

In other words, the population of groups can be decomposed into sub-populations, each of which is displaying behavior across treatments that is consistent with strategic complementarity.

Note that we make no restrictions on how groups are formed. So the model allows for some match-ups to be more common than others.

But we do require the distribution of group types to be the same across treatments.

Cross sectional data sets

Definition: A dataset $\mathcal{O} = \{\mu^t, A^t, y^t\}_{t \in \mathcal{T}}$ is consistent with strategic complementarity if there exists a probability distribution Q on A^*, the set of SC-consistent paths in the set of environments $E = \{(A^t, y^t)\}_{t \in \mathcal{T}}$, such that, for each $a \in A^t$ and $t \in \mathcal{T}$,

$$\mu^t(a) = \sum_{\hat{a} \in A^*} Q(\hat{a}) \mathbf{1}(\hat{a}^t = a).$$

By our main result (Theorem 1), we obtain the following.

Corollary: A dataset $\mathcal{O} = \{\mu^t, A^t, y^t\}_{t \in \mathcal{T}}$ is consistent with strategic complementarity if and only if there exists a probability distribution Q on A^*, the set of ARC-consistent paths on the set of environments $E = \{(A^t, y^t)\}_{t \in \mathcal{T}}$, such that, for each $a \in A^t$ and $t \in \mathcal{T}$,

$$\mu^t(a) = \sum_{\hat{a} \in A^*} Q(\hat{a}) \mathbf{1}(\hat{a}^t = a).$$

So we 'only' need to solve a set of linear equations.

Smoking behavior in marriages

We apply our tests to data on tobacco use amongst married couples.

Data on 5300+ couples from the U.S. "Current Population Survey": their smoking behavior and smoking policies at their workplace.

We focus on years 1992-1993 where there was still significant variation in workplace smoking policies.

We model each married couple as a group. The smoking decision depends on the partner's smoking decision and the smoking policy at work (permitted or otherwise).

Smoking behavior in marriages

We apply our tests to data on tobacco use amongst married couples.

Data on 5300+ couples from the U.S. "Current Population Survey": their smoking behavior and smoking policies at their workplace.

We focus on years 1992-1993 where there was still significant variation in workplace smoking policies.

We model each married couple as a group. The smoking decision depends on the partner's smoking decision and the smoking policy at work (permitted or otherwise).

Does the preference over smoking display single crossing differences, with respect to spouse's smoking choice and workplace policy? For example,

$$(S;\ N, 0) \succsim_i (N; N, 0) \Longrightarrow (S;\ S, 0) \succ_i (N; S, 0)$$

(0 means that smoking is prohibited at workplace) and

$$(S;\ N, 0) \succsim_i (N; N, 0) \Longrightarrow (S;\ S, 1) \succ_i (N; S, 1).$$

Smoking behavior in marriages

Workplace Smoking policy = (0,1)		
Wife/Husband	Non-smoking (N)	Smoking (S)
Smoking (S)	μ (N,S)=8.6%	μ (S,S)=14.9%
Non-smoking (N)	μ (N,N)=64.6%	μ (S,N)=11.9%

Workplace Smoking policy = (1,1)		
Wife/Husband	Non-smoking (N)	Smoking (S)
Smoking (S)	μ (N,S)=11.3%	μ (S,S)=16.3%
Non-smoking (N)	μ (N,N)=59.0%	μ (S,N)=13.4%

Workplace Smoking policy = (0,0)		
Wife/Husband	Non-smoking (N)	Smoking (S)
Smoking (S)	μ (N,S)=6.9%	μ (S,S)=9.4%
Non-smoking (N)	μ (N,N)=71.8%	μ (S,N)=11.9%

Workplace Smoking policy = (1,0)		
Wife/Husband	Non-smoking (N)	Smoking (S)
Smoking (S)	μ (N,S)=9.1%	μ (S,S)=14.9%
Non-smoking (N)	μ (N,N)=59.1%	μ (S,N)=16.9%

Formally, this is a data set with four treatments/observations, corresponding to the four outcomes of the exogenous variable.

$(0, 1)$ means that smoking is prohibited in the husband's workplace but not the wife's, $(1, 1)$ when it is permitted at both workplaces, etc.

Smoking behavior in marriages

Workplace Smoking policy = (0,1)		
Wife/Husband	Non−smoking (N)	Smoking (S)
Smoking (S)	μ (N,S)=8.6%	μ (S,S)=14.9%
Non−smoking (N)	μ (N,N)=64.6%	μ (S,N)=11.9%

Workplace Smoking policy = (1,1)		
Wife/Husband	Non−smoking (N)	Smoking (S)
Smoking (S)	μ (N,S)=11.3%	μ (S,S)=16.3%
Non−smoking (N)	μ (N,N)=59.0%	μ (S,N)=13.4%

Workplace Smoking policy = (0,0)		
Wife/Husband	Non−smoking (N)	Smoking (S)
Smoking (S)	μ (N,S)=6.9%	μ (S,S)=9.4%
Non−smoking (N)	μ (N,N)=71.8%	μ (S,N)=11.9%

Workplace Smoking policy = (1,0)		
Wife/Husband	Non−smoking (N)	Smoking (S)
Smoking (S)	μ (N,S)=9.1%	μ (S,S)=14.9%
Non−smoking (N)	μ (N,N)=59.1%	μ (S,N)=16.9%

Formally, this is a data set with four treatments/observations, corresponding to the four outcomes of the exogenous variable.

$(0, 1)$ means that smoking is prohibited in the husband's workplace but not the wife's, $(1, 1)$ when it is permitted at both workplaces, etc.

There are $4^4 = 256$ paths, of which 64 are ARC-consistent.

Smoking behavior in marriages

In fact, the data set fails the exact test.

Workplace Smoking policy = (0,1)		
Wife/Husband	Non–smoking (N)	Smoking (S)
Smoking (S)	μ (N,S)=8.6%	μ (S,S)=14.9%
Non–smoking (N)	μ (N,N)=64.6%	μ (S,N)=11.9%

Workplace Smoking policy = (1,1)		
Wife/Husband	Non–smoking (N)	Smoking (S)
Smoking (S)	μ (N,S)=11.3%	μ (S,S)=16.3%
Non–smoking (N)	μ (N,N)=59.0%	μ (S,N)=13.4%

Workplace Smoking policy = (0,0)		
Wife/Husband	Non–smoking (N)	Smoking (S)
Smoking (S)	μ (N,S)=6.9%	μ (S,S)=9.4%
Non–smoking (N)	μ (N,N)=71.8%	μ (S,N)=11.9%

Workplace Smoking policy = (1,0)		
Wife/Husband	Non–smoking (N)	Smoking (S)
Smoking (S)	μ (N,S)=9.1%	μ (S,S)=14.9%
Non–smoking (N)	μ (N,N)=59.1%	μ (S,N)=16.9%

Note that $\mu(N,S|1,0) = 9.1\% > 8.6\% = \mu(N,S|0,1)$.

This is impossible: any couple who chooses (N,S) under smoking policies $(1,0)$ must choose the same under $(0,1)$.

$(N,S|1,0) \succsim_h (S,S|1,0) \Rightarrow (N,S|0,1) \succ_h (S,S|0,1)$, not (S,S).

$(N,S|1,0) \succsim_h (S,S|1,0) \Rightarrow (N,N|0,1) \succ_h (S,N|0,1)$, not (S,N).

$(N,S|1,0) \succsim_w (N,N|1,0) \Rightarrow (N,S|0,1) \succ_w (N,N|0,1)$, not (N,N).

Smoking behavior in marriages

Of course the data collected is not the population distribution as such, just the outcome of a large sample.

We follow a sample inference procedure recently devised by Kitamura and Stoye (2013). They used their procedure to test the McFadden-Richter model of utility-maximization.

The test assumes that the closest consistent distribution (as measured by least squares) is the true population distribution and uses a bootstrap procedure to calculate the likelihood of obtaining, as a sample, the data we collect.

By this procedure, the probability of getting our sample (or a more extreme one) is 0.3795.

So the model cannot be rejected at a significance level of 5 or 10%.

Conclusion

We have shown how strategic complementarity can be tested on both panel data and cross sectional data.

The crucial property we need to check is the axiom of revealed complementarity (ARC).

The tests can also be modified to make data-consistent predictions for a given treatment (A^0, y^0).

In the case of panel data, Nash equilibrium predictions take the form of a subset in A^0 that (almost) has a largest and a smallest element.

In the case of cross sectional data, predictions take the form of a set of probability distributions over A^0.