Aggregating the single crossing property: theory and applications

John K.-H. Quah and Bruno Strulovici

Overview

In this talk, I will

[1] define the single crossing property,

[2] explain its application to monotone comparative statics,

[3] explain why the aggregation of this property is an important issue, using examples from optimization problems under uncertainty, and

[4] present the aggregation results in my paper with Bruno Strulovici.

John Quah
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Single crossing differences

Definition: Let $(S, \geq)$ be a poset and $\phi : S \to \mathbb{R}$. Then ϕ has the single crossing property [is a single crossing ($\mathcal{SC}$) function] if

$$\phi(s') \geq (>)\, 0 \Longrightarrow \phi(s'') \geq (>)\, 0 \text{ where } s'' > s'.$$

Definition: Let $X \subset \mathbb{R}$, S a poset, and $\{v(\cdot, s)\}_{s \in S}$ a family of functions where each $v(\cdot, s)$ is a map from X to $\mathbb{R}$.

The family $\{v(\cdot, s)\}_{s \in S}$ obeys single crossing differences if for all $x'' > x'$, the function

$$\delta(s) = v(x''; s) - v(x'; s) \text{ is an } \mathcal{SC} \text{ function.}$$

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Theorem 1: (Milgrom-Shannon) The family $\{v(\cdot; s)\}_{s \in S}$ obeys single crossing differences if and only if $\arg\max_{x \in Y} f(x; s)$ is increasing in s for all $Y \subseteq X$.

One-dimensional comparative statics

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Proof: Assume $s'' > s'$ and $x'' \in \arg\max_{x \in Y} f(x; s'')$, and $x' \in \arg\max_{x \in Y} f(x; s')$. We have to show that
$\max\{x', x''\} \in \arg\max_{x \in Y} f(x; s'')$ and
$\min\{x', x''\} \in \arg\max_{x \in Y} f(x; s')$.
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We need only consider the case where $x' > x''$.

Since $x' \in \arg\max_{x \in Y} f(x; s')$, we have $f(x'; s') \geq f(x''; s')$. By single crossing differences, $f(x'; s'') \geq f(x''; s'')$ so $x' \in \arg\max_{x \in Y} f(x; s'')$.

Furthermore, $f(x'; s') = f(x''; s')$ so that $x'' \in \arg\max_{x \in Y} f(x; s')$. If not, $f(x'; s') > f(x''; s')$ which implies (by single crossing differences) that $f(x'; s'') > f(x''; s'')$, contradicting the assumption that $f(\cdot; s'')$ is maximized at x''.

Necessity: follows from definition of single crossing differences. **QED**

One-dimensional comparative statics

Application: Consider a firm that maximizes profit

$$\Pi(x, -c) = xP(x) - cx.$$

Since $\frac{\partial^2 \Pi}{\partial x \partial c} = -1$, the family $\{\Pi(\cdot, -c)\}_{c \in R_+}$ obeys increasing (hence single crossing) differences.

More pedantically, for any $x'' > x'$,

$$\delta(-c) = x' P(x'') - x' P(x') - c(x'' - x')$$

is increasing in $-c$, hence an $\mathcal{SC}$ function.

The Milgrom-Shannon Theorem tells us that $\arg\max_{x \geq 0} \Pi(x, -c)$ is increasing in $-c$; in other words,
profit-maximizing output is decreasing in marginal cost.

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profit-maximizing output is decreasing in marginal cost.

What if there is uncertainty in c, and output has to be chosen before c is realized?

Optimization under uncertainty - Problem I

The agent chooses x to maximize

$$V(x, \theta) = \int_S v(x, s)\lambda(s, \theta)\, ds$$

where $\lambda(\cdot, \theta)$ is the density function over $s \in S \subset \mathbb{R}$ (one-dimensional uncertainty).

Suppose $\{v(\cdot; s)\}_{s \in S}$ obeys single crossing differences; then the optimal action is increasing in the state if the state is known.

If the state is uncertain, we would still expect the optimal action to be higher if higher states are more likely.

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Definition: $\{\lambda(\cdot,\theta)\}_{\theta\in\Theta}$ obeys the monotone likelihood ratio order if

$$\frac{\lambda(s,\theta'')}{\lambda(s,\theta')} \text{ is increasing in } s \text{ whenever } \theta'' > \theta'.$$

Optimization under uncertainty - Problem I

Theorem 2: Let $S \subset \mathbb{R}$ and suppose $\delta : S \to \mathbb{R}$ is an $\mathcal{SC}$ function and $\{\lambda(\cdot, \theta)\}_{t \in \Theta}$ obeys the MLR order. Then

$$\Delta(\theta) = \int_S \delta(s)\lambda(s, \theta)\, ds \text{ is an } \mathcal{SC} \text{ function (of } \theta).$$

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Corollary 1: Suppose that $\{v(\cdot; s)\}_{s \in S}$ obeys single crossing differences and $\{\lambda(\cdot, \theta)\}_{\theta \in \Theta}$ obeys the monotone likelihood ratio order. Then $\{V(\cdot; s)\}_{\theta \in \Theta}$ obeys single crossing differences, where

$$V(x, \theta) = \int_S v(x, s)\lambda(s, \theta)\, ds.$$

Consequently, $\arg\max_{x \in X} V(x; \theta)$ is increasing in θ.

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Consequently, $\arg\max_{x\in X} V(x; \theta)$ is increasing in θ.

Proof: Note that

$$\Delta(\theta) = V(x'', \theta) - V(x', \theta) = \int_S [v(x'', s) - v(x', s)]\lambda(s, \theta)\, ds = \int_S \delta(s)\lambda(s, \theta)\, ds$$

Since $\{v(\cdot; s)\}_{s\in S}$ obeys single crossing differences, δ is an $\mathcal{SC}$ function. Conclusion follows immediately Theorem 2. **QED**

Optimization under uncertainty - Problem I

Proof of Theorem 2: Let $\theta'' > \theta'$. We split $\Delta(\theta'') = \int_S \delta(s)\lambda(s,\theta'')\,ds$ into two parts:

$$\Delta(\theta'') = \int_{-\infty}^{s_0} \delta(s)\lambda(s,\theta')\,\frac{\lambda(s,\theta'')}{\lambda(s,\theta')}\,ds + \int_{s_0}^{\infty} \delta(s)\lambda(s,\theta')\,\frac{\lambda(s,\theta'')}{\lambda(s,\theta')}\,ds,$$

where $\delta(s) \leq 0$ for $s < s_0$ and $\delta(s) > 0$ for $s > s_0$. The first term on the right is greater than

$$\frac{\lambda(s_0,\theta'')}{\lambda(s_0,\theta')}\int_{-\infty}^{s_0} \delta(s)\lambda(s,\theta')\,ds$$

while the second term is greater than

$$\frac{\lambda(s_0,\theta'')}{\lambda(s_0,\theta')}\int_{s_0}^{\infty} \delta(s)\lambda(s,t_1)\,ds.$$

Adding up the two lower bounds gives us

$$\Delta(\theta'') \geq \frac{\lambda(s_0,\theta'')}{\lambda(s_0,\theta')}\int_S \delta(s)\lambda(s,\theta')\,ds = \frac{\lambda(s_0,\theta'')}{\lambda(s_0,\theta')}\Delta(\theta').$$

So $\Delta(\theta') \geq (>)\, 0$ implies $\Delta(\theta'') \geq (>)\, 0$. **QED**

Optimization under uncertainty - Problem I

Application: Consider a firm that maximizes profit

$$\Pi(x, -c) = xP(x) - cx.$$

We have shown that $\arg\max_{x \geq 0} \Pi(x, -c)$ is increasing in $-c$.

Now suppose that the firm has to choose x *before* c is known. Given its Bernoulli utility function u, the firm's objective function is

$$V(x;t) = \int u(\Pi(x, -c))\lambda(c, \theta)\, dc$$

where $\lambda(\cdot, \theta)$ is a density function (defined over c).
Note that $v(x; -c) \equiv u(\Pi(x, -c))$ obeys single crossing differences:

$$\delta(-c) = u(\Pi(x'', -c)) - u(\Pi(x', -c))$$

is a single crossing function of $-c$ (for any $x'' > x'$).

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Theorem 2 says that when higher c becomes more likely (in the MLR sense), then the firm will choose to produce less.

Optimization under uncertainty - Problem II

Problem:[a] Suppose that the firm faces uncertainty, not in its cost, but in its inverse demand function. Formally, it's problem is to choose x to maximize

$$V(x, -c) = \int_T u(xP(x,t) - cx)\lambda(t)\, dt.$$

When is it still true that higher c leads to lower output?

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More generally, suppose that for every $t \in T$, the family $\{v(\cdot, s, t)\}_{s\in S}$ obeys single crossing differences (in (x, s)). By M-S Theorem, $\arg\max_{x\in X} v(x, s, t)$ is increasing in s.

To guarantee that $\arg\max_{x\in X} V(x, s)$ is increasing in s, where

$$V(x,s) = \int_T v(x,s,t)\lambda(t)\,dt$$

it suffices that the family $\{V(\cdot, s)\}_{s\in S}$ obey single crossing differences.

This is a different and harder problem from Problem I.

[a] From this point on, slides cover results in *Aggregating the single crossing property*, by Q & S.

Optimization under uncertainty - Problem II

By definition, $V(x,s) = \int_T v(x,s,t)\lambda(t)\,dt$ obeys single crossing differences if for all $x'' > x'$,

$$\Delta(s) = \int_T [v(x'',s,t) - v(x',s,t)]\lambda(t)\,dt$$

is an $\mathcal{SC}$ function.

Define $\delta(s,t) = v(x'',s,t) - v(x',s,t)$, so

$$\Delta(s) = \int_T \delta(s,t)\lambda(t)dt$$

For each t, $\delta(\cdot,t)$ is an $\mathcal{SC}$ function (of s) because $\{v(\cdot,s,t)\}_{s\in S}$ obeys single crossing differences (in (x,s)).

So the issue is effectively this: when is the weighted sum of $\mathcal{SC}$ functions an $\mathcal{SC}$ function?

Summing single crossing functions

Let g and h, maps from the poset $(S, \geq)$ to $\mathbb{R}$, be $\mathcal{SC}$ functions.

Clearly, if f and g are increasing functions, then $\alpha f + \beta g$ is an $\mathcal{SC}$ function for any positive scalars α and β. But there's more...

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Proposition 1: The sum $\alpha f + \beta g$ is an $\mathcal{SC}$ function for any positive scalars α and β if and only if f and g obey signed-ratio monotonicity:

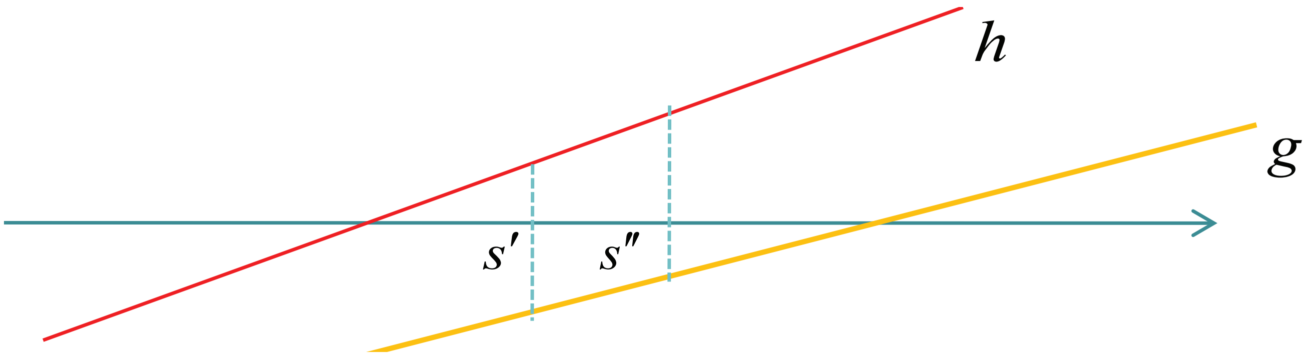
whenever $h(s')$ and $g(s')$ are of opposite signs (with no loss of generality assume that $h(s') > 0$ and $g(s') < 0$), then

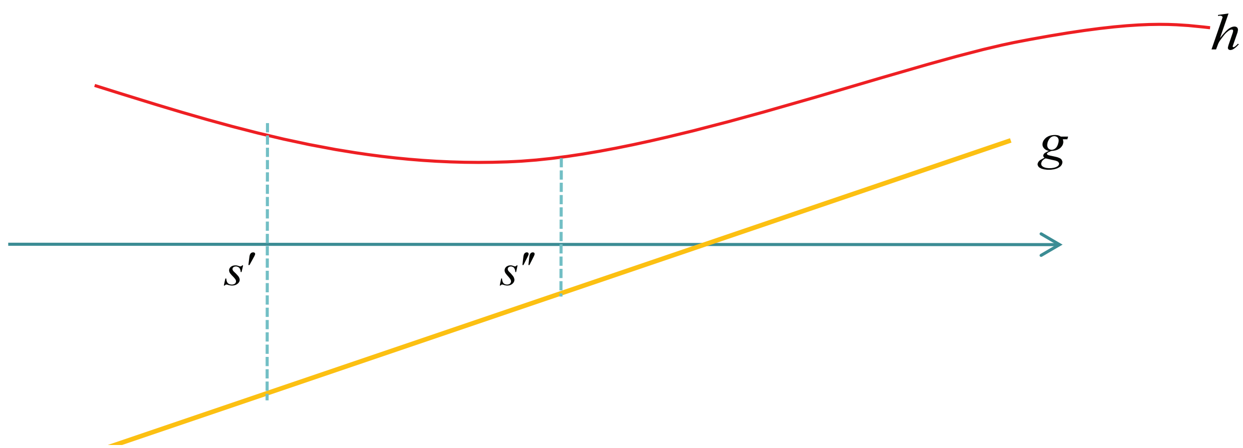
$$-\frac{g(s')}{h(s')} \geq -\frac{g(s'')}{h(s'')} \text{ whenever } s'' > s'$$

Summing single crossing functions

When $g(s') < 0$ and $h(s') > 0$, we require

$$-\frac{g(s')}{h(s')} \geq -\frac{g(s'')}{h(s'')}.$$





Summing single crossing functions

Example: Suppose $\mathbb{S} = \mathbb{R}$ and consider $h(s) = s^2 + 1$ and $g(s) = s^3$. In this case h is not an increasing function but h and g obey signed-ratio monotonicity. Clearly, h and g are of opposite signs only if $s < 0$; in that case,

$$-\frac{g(s)}{h(s)} = -\frac{s^3}{(s^2+1)}$$

is decreasing in s, for $s \in (-\infty, 0)$.

Proposition 2: Let f, g, and h be three $\mathcal{SC}$ functions that obey signed-ratio monotonicity pairwise. Then $\alpha f + \beta g + \gamma h$ is an $\mathcal{SC}$ function.

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Proposition 2: Let f, g, and h be three $\mathcal{SC}$ functions that obey signed-ratio monotonicity pairwise. Then $\alpha f + \beta g + \gamma h$ is an $\mathcal{SC}$ function.

Theorem 3: Let $\{g(\cdot; t)\}_{t \in T}$ be a family of $\mathcal{SC}$ functions that obey signed-ratio monotonicity pairwise. Then

$$G(s) = \int_T g(s;t)\lambda(t)\,dt \text{ is an } \mathcal{SC} \text{ function}$$

for any nonnegative function λ.

Optimization under uncertainty - Problem II

Corollary 2: Suppose that, for any fixed t, $v(x, s, t)$ has single crossing differences in (x, s). Then

$$V(x,s) = \int_T v(x,s,t)\lambda(t)\,dt$$

has single crossing differences if, for any $x'' > x'$ and $\tilde{t}$ and $\hat{t}$, the functions $\delta(\cdot, \tilde{t})$ and $\delta(\cdot, \hat{t})$, where

$$\delta(s,\tilde{t}) = v(x'', s, \tilde{t}) - v(x', s, \tilde{t}),$$

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$$\delta(s,\tilde{t}) = v(x'',s,\tilde{t}) - v(x',s,\tilde{t}),$$

obey signed-ratio monotonicity, i.e., if $\delta(s^*,\tilde{t}) > 0$ and $\delta(s^*,\hat{t}) < 0$, then

$$\begin{aligned}
-\frac{\delta(s^*,\hat{t})}{\delta(s^*,\tilde{t})} &= \frac{-v(x'',s^*,\hat{t}) + v(x',s^*,\hat{t})}{v(x'',s^*,\tilde{t}) - v(x',s^*,\tilde{t})} \\
&\geq \frac{-v(x'',s^{**},\hat{t}) + v(x',s^{**},\hat{t})}{v(x'',s^{**},\tilde{t}) - v(x',s^{**},\tilde{t})} \\
&= -\frac{\delta(s^{**},\hat{t})}{\delta(s^{**},\tilde{t})} \quad \text{for } s^{**} > s^*.
\end{aligned}$$

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Application: The firm faces uncertainty in its inverse demand function. Formally, it's problem is to choose x to maximize

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where $T \subset \mathbb{R}$. When is it true that higher c leads to lower output?

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where $T\subset\mathbb{R}$. When is it true that higher c leads to lower output?

Applying the corollary, $\{V(\cdot,-c)\}_{c>0}$ has single crossing differences if, for any $\tilde{t}$ and $\hat{t}$, $\delta(\cdot,\tilde{t})$ and $\delta(\cdot,\hat{t})$ obey signed-ratio monotonicity, where

$$\delta(s,t)=u(x''P(x'',t)-cx'')-u(x'P(x',t)-cx').$$

This is true if

(i) u obeys DARA
(ii) P is decreasing in x, increasing in t, and logsupermodular in (x,t).

Summing single crossing functions

$\mathcal{SC}$ functions are relevant in other contexts besides monotone comparative statics and Theorem 3 can be useful for those applications as well.

Application: A quasiconvex function defined on an interval $I \subseteq \mathbb{R}$ can be characterized as a function with a derivative that is an $\mathcal{SC}$ function.

Consider the profit function $\Pi(x) = xP(x) - C(x)$ (where x is output). This function is quasiconcave if

$$-\Pi'(x) = [-P(x)] + [-xP'(x) + C'(x)]$$

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is an $\mathcal{SC}$ function.

Suppose P is positive, decreasing, and log-concave and C is increasing and convex.

Then the maps $x \mapsto -P(x)$ and $x \mapsto -xP'(x) + C'(x)$ are $\mathcal{SC}$ functions and they obey signed-ratio monotonicity.

Hence Π is a quasiconcave function of x.

Summing single crossing functions

Theorem 3: Let $\{g(\cdot;t)\}_{t\in T}$ be a family of $\mathcal{SC}$ functions that obey signed-ratio monotonicity pairwise. Then

$$G(s) = \int_T g(s;t)\lambda(t)\,dt \text{ is an } \mathcal{SC} \text{ function}$$

for any nonnegative function λ.

When applying this theorem, T is typically the set of possible states.

If uncertainty is one-dimensional, pairwise signed-ratio monotonicity is often reasonable.

If T is multi-dimensional, signed-ratio monotonicity for all possible pairs t and t' is a strong condition.

The next theorem imposes more structure on T,
it assumes $T = \Pi_{i=1}^n T_i$, where $T_i \subseteq \mathbb{R}$,
but requires signed-ratio monotonicity only for *ordered* pairs of t and t'.

Summing single crossing functions

Theorem 4: Let $T = \Pi_{i=1}^{n} T_i$ (where $T_i \subseteq \mathbb{R}$) and $(S, \geq)$ be a poset.

Then

$$F(s) = \int_T f(s,t)\, dt$$

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is an $\mathcal{SC}$ function of s if $f(s,t)$ is an $\mathcal{SC}$ function of (s,t) and that $\forall i$, $\forall s' \in S$, and $\forall\, t'' > t'$,

(i) the functions $f(s,t')$ and $f(s,t'')$ of $s \in S$ obey signed-ratio monotonicity and
(ii) the functions $f(s', t_i, t'_{-i})$ and $f(s', t_i, t''_{-i})$ of $t_i \in T_i$ obey signed-ratio monotonicity.

Monotone Bayesian games

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Theorem: (Athey) A Bayesian game has an equilibrium in monotone strategies if the following holds:

whenever all other players are playing monotone strategies, a player has a best response that is also a monotone strategy.

Given Athey's theorem, the problem of equilibrium existence in Bayesian games reduces to establishing a comparative statics result.

Monotone Bayesian games

Consider an n-player Bayesian game where player i observes a signal $s_i \in \mathbb{S}_i \subset \mathbb{R}$. The joint density of $s = (s_1, s_2, ..., s_n)$ is given by the function λ. Player 1's posterior distribution after observing s_1 is denoted by $\lambda(s_2, s_3, ..., s_n | s_1)$.

The utility of Player 1 in state s and given actions $(x_1, x_2, ..., x_n)$ is denoted by $u((x_1, x_2, ..., x_n); s)$.

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Suppose that for $i \geq 2$, player i has strategy ϕ_i, i.e., player i takes action $\phi_i(s_i) \in X_i$ after observing s_i.

Therefore, Player 1's expected utility if he takes action $x \in X_1$ after observing s_1 is

$$\int_{\mathbb{S}_n} \dots \int_{\mathbb{S}_3} \int_{\mathbb{S}_2} u\left((x, \phi_2(s_2), \phi_3(s_3), ..., \phi_n(s_n))\,; s\right) \lambda(s_2, ..., s_n | s_1)\, ds_2 ... ds_n.$$

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Suppose that ϕ_i is increasing for all $i \geq 2$; under what conditions will Player 1 also have an increasing strategy?

Optimization under uncertainty - Problem III

After observing s_1, Player 1 chooses x to maximize

$$V(x;s_1)=\int_{\mathbb{S}_{-1}} v(x;s_1,s_2,...,s_n)\lambda(s_2,s_3,...,s_n|s_1)ds_{-1}$$

where $v(x;s)=u\left((x,\phi_2(s_2),\phi_3(s_3),...,\phi_n(s_n))\,;s_1,s_2,...,s_n\right).$

This is a problem with multi-dimensional uncertainty.

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This is a problem with multi-dimensional uncertainty.

We know that $\arg\max_{x \in X} V(x; s_1)$ increases with s_1 if, for $x'' > x'$,

$$\begin{aligned} \Delta(s_1) &= V(x''; s_1) - V(x'; s_1) \\ &= \int_{\mathbb{S}_{-1}} [v(x''; s) - v(x'; s)]\lambda(s_2, s_3, ..., s_n|s_1)\, ds_{-1} \\ &= \int_{\mathbb{S}_{-1}} \delta(s_1, s_2, ..., s_n)\lambda(s_2, s_3, ..., s_n|s_1)\, ds_{-1}. \end{aligned}$$

is an $\mathcal{SC}$ function of s_1.

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is an $\mathcal{SC}$ function of s_1. Theorem 4 can be applied to this problem, provided the map $(s_1, s_{-1}) \mapsto \delta(s_1, s_{-1})\lambda(s_{-1}|s_1)$ obeys the required properties.

Optimization under uncertainty - Problem III

Application: Consider a Bertrand oligopoly with n firms.

Firm 1 has a constant unit cost of production of c_1. The demand for its output is $D(p_1, p_{-1}; s_1)$, where s_1 is some parameter affecting demand that is observed by firm 1 and $p_{-1} = (p_2, p_3, ..., p_n)$ are the prices charged by other firms.

At prices $p = (p_1, p_{-1})$ and the parameter s_1, firm 1's profit is

$$\Pi(p_1, p_{-1}; s_1) = (p_1 - c_1)D(p_1, p_{-1}; s_1).$$

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Suppose that firm $j \neq 1$ charges the price $\psi_j(s_j)$ whenever it observes s_j. If so, Firm 1 chooses p_1 to maximize its expected utility

$$V(p_1; s_1) = \int_{\mathbb{S}_{-1}} u(\Pi(p_1, [\psi_j(s_j)]_{j\neq 1}; s_1))\lambda(s_{-1}|s_1)\, ds_{-1},$$

where u is the firm's Bernoulli utility function and $\lambda(\cdot|s_1)$ is the distribution of s_{-1}, conditional on observing s_1.

Optimization under uncertainty - Problem III

Firm 1 chooses p_1 to maximize interim expected utility

$$V(p_1; s_1) = \int_{\mathbb{S}_{-1}} u(\Pi(p_1, [\psi_j(s_j)]_{j \neq 1}; s_1))\lambda(s_{-1}|s_1)\, ds_{-1}.$$

$\arg\max_{p_1 \geq 0} V(p_1; s_1)$ is increasing in s_1 if, for all $p_1'' > p_1'$,

$$\Delta(s_1) = V(p_1''; s_1) - V(p_1'; s_1)$$

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Optimization under uncertainty - Problem III

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$$\Delta(s_1) = V(p_1''; s_1) - V(p_1'; s_1)$$

is an $\mathcal{SC}$ function of s_1. $\Delta(s_1) = \int_{S_{-1}} \delta(s_1, s_{-1})\, ds_{-1}$, where

$$\delta(s_1, s_{-1}) = [u(\Pi(p_1'', [\psi_j(s_j)]_{j\neq 1}; s_1)) - u(\Pi(p_1', [\psi_j(s_j)]_{j\neq 1}; s_1))]\ \lambda(s_{-1}|s_1).$$

If the signals $s = (s_1, s_2, ..., s_n)$ are affiliated and various conditions on risk aversion and price elasticities are jointly satisfied, then δ has the properties required by Theorem 4.

It follows that Δ is an $\mathcal{SC}$ function and, hence, $\arg\max_{p_1\geq 0} V(p_1; s_1)$ is increasing in s_1.