REVEALED PRICE PREFERENCE: THEORY AND STOCHASTIC TESTING

RAHUL DEB, YUICHI KITAMURA, JOHN K.-H. QUAH, AND JÖRG STOYE[♢]
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ABSTRACT. We develop a model of demand where consumers trade-off the utility of consumption against the disutility of expenditure. This model is appropriate whenever a consumer's demand over a *strict* subset of all available goods is being analyzed. Data sets consistent with this model are characterized by the absence of revealed preference cycles over prices. The model is readily generalized to the random utility setting, for which we develop nonparametric statistical tests. Our application on national household consumption data provides support for the model.

1. INTRODUCTION

Imagine a consumer who is asked what quantity she will purchase of L goods at given prices; in formal terms, she is asked to choose a bundle $x^t \in \mathbb{R}^L_+$ at the price vector $p^t \in \mathbb{R}^L_{++}$. To fix ideas, we could think of these goods as grocery items and x^t as the monthly purchase of groceries if p^t are the prevailing prices. With T such observations, the data set collected is $\mathcal{D} = \{(p^t, x^t)\}_{t=1}^T$. What patterns of choices in $\mathcal{D}$ should we expect to observe?

There are at least two ways to approach this question. If, at observations t and t', we find that $p^{t'} \cdot x^t < p^{t'} \cdot x^{t'}$, then

the consumer has revealed that she strictly prefers the bundle $x^{t'}$ to the bundle x^t.

If this were not true, then either x^t or a nearby bundle would be strictly preferred to $x^{t'}$ and cost less, which means the choice of $x^{t'}$ is not optimal. The standard revealed preference theory of consumer demand is built on requiring that this preference over grocery bundles, as revealed by a data set such as $\mathcal{D}$, is internally consistent. In particular, Afriat (1967)'s Theorem says that so long as the consumer's revealed preferences over bundles do not contain cycles (a property known as the *generalized axiom of revealed preference* or GARP, for short) then there is a strictly increasing utility function $U : \mathbb{R}^L_+ \to \mathbb{R}$ such that x^s maximizes $U(x)$ in the budget set $\{x \in \mathbb{R}^L_+ : p^s \cdot x \leq p^s \cdot x^s\}$,

[♢]UNIVERSITY OF TORONTO, YALE UNIVERSITY, JOHNS HOPKINS UNIVERSITY AND BONN UNIVERSITY.
E-mail addresses: rahul.deb@utoronto.ca, yuichi.kitamura@yale.edu, john.quah@economics.ox.ac.uk, stoye@uni-bonn.de.
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for every observation $s = 1, 2, ..., T$.[1] Notice that this theory implicitly assumes that it makes sense to speak of the consumer's preference over groceries, independently of her consumption of other goods, currently or in the future. In formal terms, this requires that the consumer has a preference over grocery bundles that is *weakly separable* from her consumption of all other goods.[2]

But, presented with the same data set $\mathcal{D} = \{(p^t, x^t)\}_{t=1}^T$, it would be entirely natural for us (as observers) to think along different lines; instead of inferring the consumer's preference over grocery bundles from the observations, we could draw conclusions about the consumer's preference over prices. If at observations t and t', we find that $p^{t'} \cdot x^t < p^t \cdot x^t$, then

the consumer has revealed that she strictly prefers the price $p^{t'}$ to the price p^t.

This is because, at the price vector $p^{t'}$ the consumer can, if she wishes, buy the bundle bought at the price p^t and would still have money left for the purchase of other, non-grocery, consumption goods. (Put another way, if p^t and $p^{t'}$ are the prices at two grocery stores and the consumer could choose to go to one store or the other, then the observations in $\mathcal{D}$ will lead us to conclude that she will opt for the store where the prices are $p^{t'}$.) This concept of revealed preference recognizes that there are alternative uses to money besides groceries and that expenditure on groceries has an opportunity cost. Is it possible to build a revealed preference theory of consumer demand based on this alternate requirement that the consumer's preference over grocery prices, as revealed by the data set $\mathcal{D}$, is internally consistent; if so, what would such a theory look like? The objective of this paper is to answer this question and to demonstrate the appealing features of a theory of revealed price preference.

1.1. *The expenditure-augmented utility model*

We show that the absence of revealed preference cycles over prices — a property we call the *generalized axiom of revealed price preference* or GAPP, for short — has a very natural characterization. It is both necessary and sufficient for the existence of a strictly increasing function $U : \mathbb{R}^L_+ \times \mathbb{R}_- \to \mathbb{R}$ such that $x^s \in \arg\max_{x \in \mathbb{R}^L_+} U(x, -p^s \cdot x)$ for all $s = 1, 2, ..., T$. The function U should be interpreted as an *expenditure-augmented utility function*, where $U(x, -e)$ is the consumer's utility when she acquires x at the cost of e. Unlike the standard consumer optimization problem, notice that the consumer does not have a budget constraint; instead, she is deterred from choosing arbitrarily large bundles by the increasing expenditure it incurs, which reduces her utility. This is a reduced form utility function which implicitly holds fixed all other variables that may be relevant to the consumer's preference over $(x, -e)$; these variables could include the consumer's wealth, the prices of other goods which the consumer considers relevant to her consumption decision on these L goods, etc.

[1] The term GARP is from Varian (1982), which also contains a proof of the result. An extension of Afriat's Theorem to nonlinear budget sets can also be found in Forges and Minelli (2009).

[2] The consumer's overall utility function will have the form $V(U(x), z)$, where U is the utility function defined over grocery bundles x, z is the bundle of all other goods consumed by the consumer and V is the overall utility function.

Besides being behaviorally compelling in its own right, the expenditure-augmented utility model has two distinctive features that makes it a worthwhile alternative to the standard model. (i) Notice that the marginal rate of substitution between two goods at a given bundle $x \in \mathbb{R}_+^L$ will typically depend on the expenditure incurred in acquiring that bundle; it follows that the marginal rate of substitution can vary with unobserved goods (whose consumption levels could change as e changes). In other words, our model does not require the assumption of weak separability and so it could be appropriate is situations where that assumption is problematic. (ii) Essentially because the standard model only tries to model the agent's preference among bundles of the L goods, the only price information it requires are the prices of those goods *relative to each other*: it does not require the researcher to have any information on the prices of any other good that the consumer may also purchase. This modest informational requirement is an important advantage but it also means that the model cannot tell us anything about the consumer's *overall* welfare when the prices for the L goods change. On the other hand our model recognizes that expenditure levels are endogenous (i.e., chosen by the consumer at the observed prices) and exploits this to compare the consumer's welfare at different prices for the L goods; in fact, as we have pointed out, the model is *characterized* by this feature.

In the theoretical literature in public economics and industrial organization, it is common to assume individuals have quasilinear utility functions. In addition to tractability, this assumption ensures that the difference in consumer surplus due to price changes is an exact measure of the change in welfare (as it coincides with the compensating and equivalent variations) because quasilinearity ensures that there are no income effects.[3] This is a special case of our model, with $U(x, e) = H(x) - e$ for some real-valued and increasing function H. Our approach allows greater flexibility in the form of the utility function U and, in particular, does not require a constant marginal disutility of expenditure, which imposes strong and sometimes implausible restrictions on demand.

In the empirical literature, welfare changes are often calculated by explicitly introducing a numeraire good over and above the L goods being examined and then imputing demand for this numeraire good by using data on (for example) annual income; one could then calculate the impact on welfare following prices changes on these L goods at a given income and a given price for the numeraire good.[4] Obviously, compared with this approach, ours is useful when information on income is not available (which is a feature of many commonly used data sets). But even when this information is available, by not using it, we are avoiding taking a stand on the precise budget from which the consumer draws her expenditure on these L goods; for example, the consumer could mentally set aside some expenditure for a group of commodities containing these L goods and this 'mental budget' may differ from the annual income (see Thaler (1999)). That said, our approach does rely on the assumption that consumer's utility function U is stable over the period

[3]See, for instance, Varian (1985), Schwartz (1990) and the papers that follow.

[4]For recent work using this approach, see Blundell, Horowitz, and Parey (2012) and Hausman and Newey (2016). In both these papers, the empirical application involves the case where $L = 1$ (specifically, the good examined is gasoline, so it is a two-good demand system when the numeraire is included); when $L = 1$, our approach imposes no meaningful restrictions on data so it does not readily provide a viable alternative way to study the specific empirical issues in those papers.

where her demand is observed; presumably large changes to the consumer's wealth (both her current resources and her future prospects) will have an impact on U, so in effect we are assuming that these fluctuations are modest.

1.2. *Testing the expenditure-augmented utility model*

After formulating the expenditure-augmented utility model and exploring its theoretical features, the second part of our paper is devoted to testing it empirically. We could in principle test this model on panel data sets of household or individual demand, for example, from information on purchasing behavior collected from scanner data. Tests and applications of the standard model using Afriat's Theorem or its extensions are very common,[5] and our model could be tested on this data in the same fashion; GAPP is just as straightforward to test as GARP on a panel data set.[6] Rather than doing this, we develop a random utility version of our model and implement an econometric test of this version instead. There are at least three reasons why it is useful to develop such a model. (i) The random utility model is more general in that it allows for individual preferences in the population to change over time, provided the population distribution of preferences stays the same; this weaker assumption may be appropriate for data sets that span longer time horizons. (ii) The random utility version can be applied to repeated cross sectional data and does not require panel data sets. (iii) Perhaps most importantly, nonparametric statistical procedures could be naturally introduced to test the random utility model and so we can go beyond the binary pass-fail tests of the nonstochastic expenditure augmented-utility model.

McFadden and Richter (1991) have formulated a random utility version of the standard model that could be used to test data collected from repeated cross sections. They assume that the econometrician observes the distribution of demand choices on each of a finite number of budget sets and characterize the observable content of this model, under the assumption that the distribution of preferences is stable across observations. Their key observation is that, notwithstanding the potentially infinite number of possible preference types generating the observed demand distributions, it is possible to partition this infinite set to a *finite* number of equivalence classes with the members of each class generating demand conforming to a particular pattern.[7] This in turn guarantees that the observations are rationalizable by a random version of the standard model if and only if there is a solution to a linear program constructed from the data.

Unfortunately, the process of actually applying the test devised by McFadden and Richter is in fact rather complicated, simply because observational data does not typically come in the form they have assumed. Indeed a population of consumers will have different expenditure levels at the same prices and, what is worse, these expenditure levels are chosen by the consumers themselves, so one simply does not directly observe the distribution of demand when all consumers are subjected to the same budget set. Recently, Kitamura and Stoye (2013) have implemented a test based on the McFadden-Richter model but they could only do so by first estimating, at each

[5]For a recent survey, see Crawford and De Rock (2014).

[6]It is also common to test the standard model in experimental settings. Note that, our test is *not* appropriate on typical experimental data where the budget is provided exogenously (and is not endogenous as in our model) as part of the experiment.

[7]The number of classes is bounded above by a formula that depends on the number of distinct budget sets.

observed price vector, the distribution of demand at a common expenditure level (distinct from those expenditure levels actually observed). Nonparametric estimation of this type is not impossible, but it does require the use of instrumental variables with all its attendant assumptions.

We devise a test of the random utility version of our model and show that it is both easier to implement and requires no additional data (such as an instrumental variable) in contrast to the test of the McFadden-Richter model. We assume that the econometrician observes the distribution of demand at a finite set of price vectors. At each price vector, expenditure levels need not be common across consumers but can vary across consumers and across different price vectors, so this corresponds almost exactly to the characteristics of observational data on demand. We show that such a data set can be generated by a stable distribution of expenditure-augmented utility functions if and only if there is a solution to a family of linear inequalities constructed from the data. When a solution exists, its solution gives the proportion of consumers belonging to each of a finite number of utility classes. Through these solutions, we also obtain upper and lower bounds on the proportion of consumers who are revealed better off, or worse off, at one price vector compared to another. In short, not only can we test the stochastic version of the expenditure-augmented utility model, the test also furnishes useful information on the welfare impact of price changes on the population of consumers.

1.3. *Organization of the paper*

The remainder of this paper is structured as follows. Section 2 lays our the deterministic model and its revealed preference characterization, and Section 3 generalizes it to our analog of a Random Utility Model. Section 4 lays out the novel econometric theory needed to test this, Section 5 illustrates with an empirical application, and Section 6 concludes.

2. The Deterministic Model

The primitive in the analysis in this section is a data set of a single consumer's purchasing behavior collected by an econometrician. The econometrician observes the consumer's purchasing behavior over L goods and the prices at which those goods were chosen. In formal terms, the bundle is in $\mathbb{R}^L_+$ and the prices are in $\mathbb{R}^L_{++}$ and so an observation t can be represented as $(p^t, x^t) \in \mathbb{R}^L_{++} \times \mathbb{R}^L_+$. The *data set* collected by the econometrician is $\mathcal{D} := \{(p^t, x^t)\}_{t=1}^T$. We will slightly abuse notation and use T both to refer to the number of observations, which we assume is finite, and the set $\{1, \ldots, T\}$; the meaning will be clear from the context. Similarly, L could denote both the number, and the set, of commodities.

We shall begin with a short description of Afriat's Theorem. This provides the background to our model (especially for readers unfamiliar with that result) and, additionally, we employ it to prove some of our results.

2.1. *Afriat's Theorem*

Given a data set $\mathcal{D} := \{(p^t, x^t)\}_{t=1}^T$, a locally nonsatiated[8] utility function $\widetilde{U} : \mathbb{R}^L_+ \to \mathbb{R}$ is said to *rationalize* $\mathcal{D}$ if

$$x^t \in \underset{\{x \in \mathbb{R}^L_+ :\, p^t x \le p^t x^t\}}{\operatorname{argmax}} \widetilde{U}(x) \qquad \text{for all } t \in T. \tag{1}$$

This is the standard notion of rationalization and the one addressed by Afriat's Theorem. It requires that there exists a utility function such that each observed bundle x^t maximizes utility in the budget set given by the observed prices p^t and expenditure $p^t x^t$.

A basic concept used in Afriat's Theorem is that of *revealed preference*. This is captured by two binary relations, $\succeq_x$ and $\succ_x$, defined on the chosen bundles observed in $\mathcal{D}$, that is, the set $\mathcal{X} := \{x^t\}_{t \in T}$. These revealed preference relations are defined as follows:

$$x^{t'} \succeq_x (\succ_x) x^t \text{ if } x^{t'} p^{t'} \ge (>) x^t p^{t'}.$$

We say that the bundle $x^{t'}$ is *directly revealed preferred* to x^t if $x^{t'} \succeq_x x^t$, that is, whenever the bundle x^t is cheaper at prices $p^{t'}$ than the bundle x^t. If x^t is strictly cheaper, so $x^{t'} \succ_x x^t$, we say that $x^{t'}$ is *directly revealed strictly preferred* to x^t. This terminology is, of course, very intuitive. If the agent is maximizing some locally nonsatiated utility function $\widetilde{U}$, then if $x^{t'} \succeq_x x^t$ ($x^{t'} \succ_x x^t$), it must imply that $\widetilde{U}(x^{t'}) \ge (>) \widetilde{U}(x^t)$.

We denote the transitive closure of $\succeq_x$ by $\succeq_x^*$, that is, for $x^{t'}$ and x^t in $\mathcal{X}$, we have $x^{t'} \succeq_x^* x^t$ if there are t_1, t_2,...,t_N in T such that $x^{t'} \succeq_x x^{t_1}$, $x^{t_1} \succeq_x x^{t_2}, \ldots, x^{t_{N-1}} \succeq_x x^{t_N}$, and $x^{t_N} \succeq_x x^{t'}$; in this case, we say that $x^{t'}$ is *revealed preferred* to x^t. If anywhere along this sequence, it is possible to replace $\succeq_x$ with $\succ_x$ then we say that $x^{t'}$ is *revealed strictly preferred* to x^t and denote that relation by $x^{t'} \succ_x^* x^t$. Once again, this terminology is completely natural since if $\mathcal{D}$ is rationalizable by some locally nonsatiated utility function $\widetilde{U}$, then $x^{t'} \succeq_x^* (\succ_x^*) x^t$ implies that $\widetilde{U}(x^{t'}) \ge (>) \widetilde{U}(x^t)$.

Definition 2.1. A data set $\mathcal{D} = \{(p^t, x^t)\}_{t=1}^T$ satisfies the Generalized Axiom of Revealed Preference or **GARP** if there do not exist two observations $t, t' \in T$ such that $x^{t'} \succeq_x^* x^t$ and $x^t \succ_x^* x^{t'}$.

Afriat showed that this condition is necessary and sufficient for rationalization.

Afriat's Theorem (Afriat (1967))**.** *Given a data set $\mathcal{D} = \{(p^t, x^t)\}_{t=1}^T$, the following are equivalent:*

(1) $\mathcal{D}$ can be rationalized by a locally nonsatiated utility function
(2) $\mathcal{D}$ satisfies GARP.
(3) $\mathcal{D}$ can be rationalized by a strictly increasing, continuous, and concave utility function.

REMARK 1. The proof that the rationalizability of $\mathcal{D}$ implies GARP is straightforward to show (it essentially follows from the discussion preceding the statement of the theorem), so the heart of the theorem lies in showing that GARP implies that $\mathcal{D}$ can be rationalized. The standard proof (for instance, see Fostel, Scarf, and Todd (2004)) works by showing that a consequence of GARP is that there exist numbers ϕ^t and $\lambda^t > 0$ (for all $t \in T$) that solve the inequalities

$$\phi^{t'} \le \phi^t + \lambda^t p^t \cdot (x^{t'} - x^t) \text{ for all } t' \ne t. \tag{2}$$

[8]This means that at any bundle x and open neighborhood of x, there is a bundle y in the neighborhood with strictly higher utility.

It is then straightforward to show that

$$\widetilde{U}(x) = \min_{t \in T} \{\phi^t + \lambda^t p^t \cdot (x - x^t)\} \tag{3}$$

rationalizes $\mathcal{D}$, with the utility of the observed consumption bundles satisfying $\widetilde{U}(x^t) = \phi^t$. The function $\widetilde{U}$ is the lower envelope of a finite number of strictly increasing affine functions, and so it is strictly increasing, continuous, and concave. A remarkable feature of this theorem is that while GARP follows simply from local nonsatiation of the utility function, it is sufficient to guarantee that $\mathcal{D}$ is rationalized by a utility function with significantly stronger properties.

REMARK 2. To be precise, GARP guarantees that there is preference $\succsim$ (in other words, a complete, reflexive, and transitive binary relation) on $\mathcal{X}$ that extends the (potentially incomplete) revealed preference relations $\succeq_x^*$ and $\succ_x^*$ in the following sense: if $x^{t'} \succeq_x^* x^t$, then $x^{t'} \succsim x^t$ and if $x^{t'} \succ_x^* x^t$ then $x^{t'} \succ x^t$. One could then proceed to show (and this is less obvious) that, for any such preference $\succsim$, the inequalities (2) admit a solution with the property that $\phi^{t'} \geq (>)\phi^t$ if $x^{t'} \succsim (\succ)x^t$ (see Quah (2014)). Hence, for any preference $\succsim$ that extends the revealed preference relations, there is in turn a utility function $\widetilde{U}$ that rationalizes $\mathcal{D}$ and extends $\succsim$ (from $\mathcal{X}$ to $\mathbb{R}^L_+$) in the sense that $\widetilde{U}(x^{t'}) \geq (>)\widetilde{U}(x^t)$ if $x^{t'} \succsim (\succ)x^t$. This has implications on the inferences one could draw from the data. If $x^{t'} \not\succsim_x^* x^t$ then it is always possible to find a preference extending the revealed preference relations such that $x^t \succ x^{t'}$. Similarly, if $x^{t'} \succeq_x^* x^t$ but $x^{t'} \not\succ_x^* x^t$ then one could find a preference extending the revealed preference relations such that $x^{t'} \sim x^t$.[9] Therefore, $x^{t'} \succeq_x^* (\succ_x^*)x^t$ if and only if every locally nonsatiated utility function rationalizing $\mathcal{D}$ has the property that $\widetilde{U}(x^{t'}) \geq (>)\widetilde{U}(x^t)$.

REMARK 3. A feature of Afriat's Theorem that is less often remarked upon is that in fact $\widetilde{U}$, as given by (3), is well-defined, strictly increasing, continuous, and concave on the domain $\mathbb{R}^L$, rather than just the positive orthant $\mathbb{R}^L_+$. Furthermore,

$$x^t \in \operatorname*{argmax}_{\{x \in \mathbb{R}^L :\, p^t x \leq p^t x^t\}} \widetilde{U}(x) \qquad \text{for all } t \in T. \tag{4}$$

In other words, x^t is optimal even if $\widetilde{U}$ is extended beyond the positive orthant and x can be chosen from the larger domain. (Compare (4) with (1).) This curiosity will turn out to be rather convenient when we apply Afriat's Theorem in our proofs.

2.2. *Consistent welfare comparisons across prices*

Typically, the L goods whose demand is being monitored by the econometrician constitutes no more than a part of the purchasing decisions made by the consumer. The consumer's true budget (especially when one takes into account the possibility of borrowing and saving) is never observed and the expenditure which she devotes to the L goods is a decision made by the consumer and dependent on prices (of the L goods and possibly on the prices of other goods as well). The consumer's choice over the L goods inevitably affects what she could afford on, and therefore her consumption of, other goods not observed by the econometrician; given this, the rationalization criterion (1) used in Afriat's Theorem will only make sense under an additional assumption of

[9]We use $x^{t'} \sim x^t$ to mean that $x^{t'} \succsim x^t$ and $x^t \succsim x^{t'}$.

weak separability: the consumer has a sub-utility function $\overline{U}$ defined over the L goods and the utility function of the consumer, defined over all goods, takes the form $H(\overline{U}(x), y)$, where x is the bundle of L goods observed by the econometrician and y is the bundle of unobserved goods. Assuming that the consumer chooses (x, y) to maximize $G(x, y) = H(\overline{U}(x), y)$, subject to a budget constraint $px + qy \leq M$ (where M is her wealth and p and q are the prices of the observed and unobserved goods respectively), then, at the prices p^t, the consumer's choice x^t will obey (1) provided H is strictly increasing in either the first or second argument. This provides the theoretical motivation to test for the existence of a sub-utility function $\widetilde{U}$ that rationalizes a data set $\mathcal{D} = \{(p^t, x^t)\}_{t=1}^T$. Notice that once the weak separability assumption is in place, the background requirements of Afriat's test are very modest in the sense that it is possible for prices of the unobserved goods and the unobserved total wealth to change arbitrarily across observations, without affecting the validity of the test. This is a major advantage in applications but the downside is that the conclusions of this model are correspondingly limited to ranking different bundles among the observed goods via the sub-utility function.

Now suppose that instead of checking for rationalizability in the sense of (1), the econometrician would like to ask a different question: given a data set $\mathcal{D} = \{(p^t, x^t)\}_{t=1}^T$, can he sign the welfare impact of a price change from p^t to $p^{t'}$? Stated differently, this question asks whether he can compare the consumer's *overall* utility after a change in the prices of the L goods from p^t and $p^{t'}$, holding fixed other aspects of the economic environment that may affect the consumer's welfare, such as the prices of unobserved goods and her overall wealth. Perhaps the most basic welfare comparison in this setting can be made as follows: if at prices $p^{t'}$, the econometrician finds that $p^{t'}x^t < p^t x^t$ then he can conclude that the agent is better off at the price vector $p^{t'}$ compared to p^t. This is because, at the price $p^{t'}$ the consumer can, if she wishes, buy the bundle bought at p^t and she would still have money left over to buy other things, so she must be strictly better off at $p^{t'}$. This ranking is eminently sensible, but can it lead to inconsistencies?

Example 1. Consider a two observation data set

$$p^t = (2, 1)\,,\; x^t = (4, 0) \;\text{ and }\; p^{t'} = (1, 2)\,,\; x^{t'} = (0, 1).$$

which is depicted in Figure 1. Given that the budget sets do not even cross, we know that GARP holds. Since $p^{t'}x^t < p^t x^t$, it seems that we may conclude that the consumer is better off at prices $p^{t'}$ than at p^t; however, it is also true that $p^{t'}x^{t'} > p^t x^{t'}$, which gives the opposite conclusion.

This example shows that for an econometrician to be able to consistently compare the consumer's welfare at different prices, some restriction (different from GARP) has to be imposed on the data set. To be precise, define the binary relations $\succeq_p$ and $\succ_p$ on $\mathcal{P} := \{p^t\}_{t \in T}$, that is, the set of price vectors observed in $\mathcal{D}$, in the following manner:

$$p^{t'} \succeq_p (\succ_p) p^t \text{ if } p^{t'}x^t \leq (<) p^t x^t.$$

We say that price $p^{t'}$ is *directly revealed preferred* to p^t if $p^{t'} \succeq_p p^t$, that is, whenever the bundle x^t is cheaper at prices $p^{t'}$ than at prices p^t. If it is strictly cheaper, so $p^{t'} \succ_p p^t$, we say that $p^{t'}$ is *directly revealed strictly preferred* to p^t. We denote the transitive closure of $\succeq_p$ by $\succeq_p^*$, that is, for $p^{t'}$ and p^t in $\mathcal{P}$, we have $p^{t'} \succeq_p^* p^t$ if there are t_1, t_2,...,t_N in T such that $p^{t'} \succeq_p p^{t_1}$, $p^{t_1} \succeq_p p^{t_2}$,..., $p^{t_{N-1}} \succeq_p p^{t_N}$, and

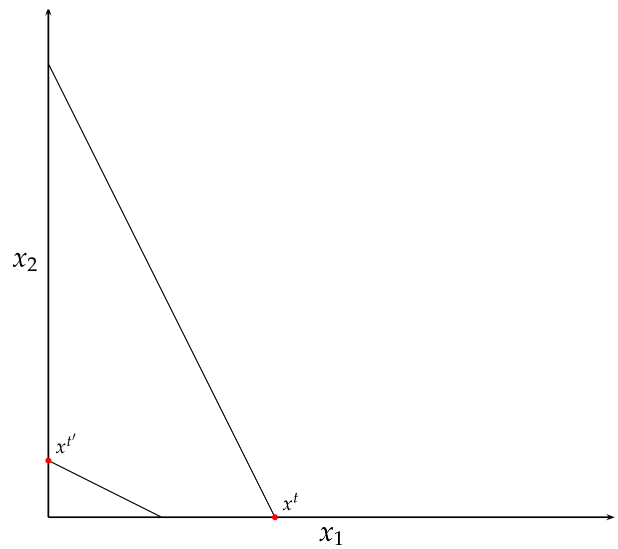


FIGURE 1. Choices that do not allow for consistent welfare predictions.

$p^{t_N} \succeq_p p^{t'}$; in this case we say that $p^{t'}$ is *revealed preferred* to p^t. If anywhere along this sequence, it is possible to replace $\succeq_p$ with $\succ_p$ then we say that $p^{t'}$ is *revealed strictly preferred* to p^t and denote that relation by $p^{t'} \succ_p^* p^t$. Then the following restriction is the bare minimum required to exclude the possibility of circularity in the econometrician's assessment of the consumer's wellbeing at different prices.

Definition 2.2. The data set $\mathcal{D} = \{(p^t, x^t)\}_{t=1}^T$ satisfies the Generalized Axiom of Price Preference or **GAPP** if there do not exist two observations $t, t' \in T$ such that $p^{t'} \succeq_p^* p^t$ and $p^t \succ_p^* p^{t'}$.

This in turn leads naturally to the following question: if a consumer's observed demand behavior obeys GAPP, what could we say about her decision making procedure?

2.3. *The Expenditure-Augmented Utility Model*

An *expenditure-augmented utility function* (or simply, an *augmented utility function*) is a utility function that has, as its arguments, both the bundle consumed by the consumer x and the total expense e incurred in acquiring the bundle. Formally, the augmented utility function U has domain $\mathbb{R}_+^L \times \mathbb{R}_-$, where $U(x, -e)$ is assumed to be strictly increasing in the last argument; in other words, utility is strictly decreasing in expenditure. This second argument captures the opportunity cost of money and is a simple, reduced form way of modeling the tradeoff with all other financial decisions made by the consumer. At a given price p, the consumer chooses a bundle x to maximize $U(x, -p \cdot x)$. We denote the *indirect utility at price p* (corresponding to U) by

$$V(p) := \sup_{x \in \mathbb{R}_+^L} U(x, -p \cdot x). \tag{5}$$

If the consumer's augmented utility maximization problem has a solution at every price vector $p \in \mathbb{R}_{++}^L$, then V is also defined at those prices and this induces a reflexive, transitive, and complete preference over prices in $\mathbb{R}_{++}^L$.

A data set $\mathcal{D} = \{(p^t, x^t)\}_{t=1}^T$ is *rationalized by an augmented utility function* if there exists such a function $U : \mathbb{R}^L_+ \times \mathbb{R}_- \to \mathbb{R}$ with

$$x^t \in \underset{x \in \mathbb{R}^L_+}{\operatorname{argmax}}\, U(x, -p^t \cdot x) \qquad \text{for all } t \in T.$$

Notice that unlike the notion of rationalization in Afriat's Theorem, we do not require the bundle x to be chosen from the budget set $\{x \in \mathbb{R}^L_+ : p^t x \leq p^t x^t\}$. The consumer can instead choose from the entire consumption space $\mathbb{R}^L_+$, though expenditure is taken into account since more costly bundles will depress the consumer's utility.

It is straightforward to see that GAPP is necessary for a data set to be rationalizable by an augmented utility function. Suppose GAPP were not satisfied but the data could be rationalized nonetheless by some augmented utility function U. Then, for some $t, t', t_1, \ldots, t_N \in T$, its indirect utility V would satisfy

$$V(p^{t'}) \geq V(p^{t_1}) \geq \cdots \geq V(p^{t_N}) \geq V(p^t) > V(p^{t'})$$

which is impossible.

Our main theoretical result, which we state next, also establishes the sufficiency of GAPP for rationalization. Moreover, the result states that whenever $\mathcal{D}$ can be rationalized, it can be rationalized by an augmented utility function U with a list of properties that make it convenient for analysis. In particular, we can guarantee that there is always a solution to $\max_{x \in \mathbb{R}^L_+} U(x, -p \cdot x)$ for any $p \in \mathbb{R}^L_{++}$. This property guarantees that U will generate preferences over all the price vectors in $\mathbb{R}^L_{++}$. Clearly, it is also necessary for making out-of-sample predictions and, indeed, it is important for the internal consistency of the model.[10]

Theorem 1. *Given a data set $\mathcal{D} = \{(p^t, x^t)\}_{t=1}^T$, the following are equivalent:*

(1) $\mathcal{D}$ can be rationalized by an augmented utility function.
(2) $\mathcal{D}$ satisfies GAPP.
(3) $\mathcal{D}$ can be rationalized by an augmented utility function U that is strictly increasing, continuous, and concave. Moreover, U is such that $\max_{x \in \mathbb{R}^L_+} U(x, -p \cdot x)$ has a solution for all $p \in \mathbb{R}^L_{++}$.

Proof. We will show that (2) $\implies$ (3). We have already argued that (1) $\implies$ (2) and (3) $\implies$ (1) by definition.

Choose a number $M > \max_t p^t x^t$ and define the augmented data set $\widetilde{\mathcal{D}} = \{(p^t, 1), (x^t, M - p^t x^t)\}_{t=1}^T$. This data set augments $\mathcal{D}$ since we have introduced an $L + 1^{\text{th}}$ good, which we have priced at 1 across all observations, with the demand for this good equal to $M - p^t x^t$.

The crucial observation to make here is that

$$(p^t, 1)(x^t, M - p^t x^t) \geq (p^t, 1)(x^{t'}, M - p^{t'} x^{t'}) \text{ if and only if } p^{t'} x^{t'} \geq p^t x^{t'},$$

which means that

$$(x^t, M - p^t x^t) \succeq_x (x^{t'}, M - p^{t'} x^{t'}) \text{ if and only if } p^t \succeq_p p^{t'}.$$

[10] Suppose the data set is rationalized but only by an augmented utility function for which the existence of an optimum is not generally guaranteed, then it undermines the hypothesis tested since it is not clear why the sample collected should then have the property that an optimum exists.

Similarly,

$$(p^t, 1)(x^t, M - p^t x^t) > (p^t, 1)(x^{t'}, M - p^{t'} x^{t'}) \text{ if and only if } p^{t'} x^{t'} > p^t x^{t'},$$

and so

$$(x^t, M - p^t x^t) \succ_x (x^{t'}, M - p^{t'} x^{t'}) \text{ if and only if } p^t \succ_p p^{t'}.$$

Consequently, $\mathcal{D}$ satisfies GAPP if and only if $\widetilde{\mathcal{D}}$ satisfies GARP. Applying Afriat's Theorem, when $\widetilde{\mathcal{D}}$ satisfies GARP, there is $\widetilde{U} : \mathbb{R}^{L+1} \to \mathbb{R}$ such that

$$(x^t, M - p^t x^t) \in \underset{\{x \in \mathbb{R}^L_+ : p^t x + m \leq M\}}{\operatorname{argmax}} \widetilde{U}(x, m) \qquad \text{for all } t \in T.$$

The function $\widetilde{U}$ can be chosen to be strictly increasing, continuous, and concave, and the lower envelope of a finite set of affine functions. Note that augmented utility function $\overline{U} : \mathbb{R}^L_+ \times \mathbb{R}_- \to \mathbb{R}$ defined by $\overline{U}(x, -e) := \widetilde{U}(x, M - e)$ rationalizes $\mathcal{D}$ as x^t solves $\max_{x \in \mathbb{R}^L_+} \overline{U}(x, -p^t \cdot x)$ by construction. Furthermore, $\overline{U}$ is strictly increasing in $(x, -e)$, continuous, and concave.

It remains to be shown that U can be maximized at all $p \in \mathbb{R}^L_{++}$. Let $h : \mathbb{R}_+ \to \mathbb{R}$ be a differentiable function with $h(0) = 0$, $h'(r) > 0$ and $h''(r) > 0$. Note that these properties guarantee that $\lim_{r \to \infty} h(r) = \infty$. Define $U : \mathbb{R}^L_+ \times \mathbb{R}_- \to \mathbb{R}$ by

$$U(x, -e) := \overline{U}(x, -e) - h(\max\{0, e - M\}). \tag{6}$$

It is clear that this function is strictly increasing in $(x, -e)$, continuous, and concave. Furthermore, x^t solves $\max_{x \in \mathbb{R}^L_+} U(x, -p^t \cdot x)$. This is because $U(x, -e) \leq \overline{U}(x, -e)$ for all $(x, -e)$, and $U(x^t, -p^t x^t) = \overline{U}(x^t, -p^t x^t)$. Lastly, we claim that at every $p \in \mathbb{R}^L_{++}$, $\operatorname{argmax}_{x \in \mathbb{R}^L_+} U(x, -p \cdot x)$ is nonempty. Given that U is continuous, this will fail to hold at some price vector p only if there is a sequence $x^n \in \mathbb{R}^L_+$ such that $p \cdot x^n \to \infty$, with $U(x^{n+1}, -px^{n+1}) > U(x^n, -px^n)$. But this is impossible because the piecewise linearity of $U(x, -e)$ in x and the strict convexity of h implies that $U(x^n, -px^n) \to -\infty$. □

From this point onwards, when we refer to 'rationalization' without additional qualifiers, we shall mean rationalization with an expenditure-augmented utility function, i.e., in the sense established by Theorem 1 rather than in the sense established by Afriat's Theorem. It is clear that the inclusion of expenditure in the augmented utility function captures the opportunity cost incurred by the consumer when she chooses some bundle of L goods. Theorem 1 says that this is precisely the form the utility function should take if we require the data set to obey GAPP, but note that the theorem does not require us to subscribe to a specific — or indeed, *any* — 'more fundamental' model from which the augmented utility function could be derived. We could, for example, interpret the augmented utility function as a representation of a preference over bundles of L goods and their associated expenditure that the consumer has developed as a habit and which guides her purchasing decisions.

That said, the proof of Theorem 1 itself provides a particular interpretation for the augmented utility function. We could suppose that the consumer is maximizing an overall utility function that depends both on the observed bundle x and on other goods y, subject to an overall wealth of M, that is, the consumer is maximizing the overall utility $G(x, y)$ subject to $px + qy \leq M$, where

q are the prices of other goods. Keeping q and M fixed, we can interpret $U(x, -e)$ as the greatest overall utility the consumer can achieve by choosing y optimally conditional on consuming x, that is,

$$U(x, -e) = \max_{\{y \geq 0 \,:\, qy \leq M - e\}} G(x, y). \tag{7}$$

It is worth repeating that a strength of our framework is that we do not have to take a stand on what goods y the consumer trades off with x or the amount M that she allocates to them. This allows us to account for potentially unobserved 'mental budgeting' (see Thaler (1999)) that the consumer may be engaged in.

The augmented utility function could be thought of as a generalization of the *quasilinear* model which is commonly employed in partial equilibrium analysis, both to model demand and also to carry out a welfare analysis of price changes. In this case, the consumer maximizes utility net of expenditure, that is, she chooses a bundle x that maximizes

$$U(x, -e) := \overline{U}(x) - e, \tag{8}$$

where $\overline{U}(x)$ is the utility of the bundle x, and $e \geq 0$ is the expense incurred when acquiring the bundle. This formulation can be motivated by assuming that the consumer has an overall utility function that is quasilinear, that is, $G(x, y) = \overline{U}(x) + y$, where there is one unobserved representative good consumed at level $y \in \mathbb{R}_+$. If we normalize the price of the outside good at 1 and assume that M is sufficiently large, then the consumer maximizes $G(x, y)$ subject to $px + y \leq M$ if and only if he chooses x to maximize $U(x, -px) = \overline{U}(x) - px$.

The quasilinear utility model imposes a strong restriction on the consumer's demand behavior that is not necessarily desirable.[11] For example, suppose $L = 2$ and the consumer at prices $(p_1, p_2) \gg (0, 0)$ prefers the bundle $(2, 1)$ to another bundle $(1, 2)$; then it is straightforward to check that this preference is maintained at the prices $(p_1 + k, p_2 + k) \gg (0, 0)$ for any k. In contrast, for the general augmented utility function, the agent's marginal rate of substitution between any two goods (among the L observed goods) can depend on the expenditure incurred in acquiring the L-good bundle.[12] So we allow for the possibility that the consumer's willingness to trade, say, food for alcohol depends on the overall expenditure incurred in acquiring the bundle; if food and alcohol prices increase and the expenditure incurred in acquiring a given bundle goes up, the consumer will have less to spend on other things such as leisure, which could well have an impact on her marginal rate of substitution between those two goods.

2.4. *Local robustness of the GAPP test*

The augmented utility function captures the idea that a consumer's choice is guided by the satisfaction she derives from the bundle as well as the opportunity cost of acquiring that bundle, as

[11]A consumer with a quasilinear utility function will generate a data set obeying both GAPP and GARP. The precise conditions on a data set that characterize rationalization with a quasilinear utility function can be found in Brown and Calsamiglia (2007).

[12]If the augmented utility function is differentiable, then the marginal rate of substitution between goods 1 and 2 is given by $(\partial U / \partial x_1)/(\partial U / \partial x_2)$ evaluated at $(x, -e)$. This will in general depend on e, though not in the quasilinear case.

measured by its expense. This opportunity cost depends on the prices of the alternative (unobserved) goods, so in empirical applications of the GAPP test (especially on data that spans long time horizons), it would make sense to deflate the prices of the observed goods by an appropriate price index (to account for price changes of the unobserved goods). Given that these indices are imperfect, the deflated prices of the observed goods p^t used for testing may well differ from their true values q^t, which introduces an observational error into the data set. Put differently, at time t, the consumer actually maximizes $U(x, -q^t x)$ instead of $U(x, -p^t x)$ which is the hypothesis we are testing.

Another potential source of error is that the agent's unobserved wealth may change from one observation to the next and this could manifest itself as a change in the consumer's stock of the representative outside good. If this happens, then at observation t, the consumer would be maximizing $U(x, -p^t x + \delta^t)$, where δ^t is the perturbation in wealth at time t, instead of maximizing $U(x, -p^t x)$. Of course, the errors could potentially enter simultaneously in both prices and wealth.

Proposition 1 below shows that inference from a GAPP test is unaffected as long as the size of the errors are bounded (equation (9) provides the specific bound). Specifically, so long as $\mathcal{D}$ obeys a mild genericity condition (which is satisfied in our empirical application), the GAPP test is locally robust in the sense that any conclusion obtained through the test remains valid for sufficiently small perturbations of the original hypothesis. For example, a data set that fails GAPP is not consistent with the maximization of an augmented utility function, for all prices sufficiently close to the ones observed and after allowing for small wealth perturbations.

Not surprisingly, if the errors are allowed to be unbounded, then the model ceases to have any content. Specifically, we can always find wealth perturbations δ^t (while prices p^t are assumed to be measured without error) such that each x^t maximizes $U(x, -p^t x + \delta^t)$, with no restrictions on $\mathcal{D}$. In particular, these perturbations can be chosen to be mean zero.

Proposition 1. *Given a data set $\mathcal{D} = \{(p^t, x^t)\}_{t=1}^T$, the following hold:*

(1) *There exists an augmented utility function U and $\{\delta^t\}_{t=1}^T$, with $\sum_{t=1}^T \delta^t = 0$, such that $x^t \in \operatorname{argmax}_{x \in \mathbb{R}^L_+} U(p^t, -p^t x^t + \delta^t)$ for all $t \in T$.*

(2) *Suppose, $\mathcal{D}$ satisfies $p^t x^t - p^{t'} x^t \neq 0$ for all $t \neq t'$ and let $\{q^t\}_{t=1}^T$ and $\{\delta^t\}_{t=1}^T$ satisfy*

$$2 \max_{t \in T} \{|\delta^t|\} + 2B \max_{t \in T, i \in L} \{|\epsilon_i^t|\} < \min_{t, t' \in T, t \neq t'} |p^t x^t - p^{t'} x^t| \tag{9}$$

where $B = \max_{t \in T} \{\sum_{i=1}^L |x_i^t|\}$ and $\epsilon_i^t := q_i^t - p_i^t$.

If $\mathcal{D}$ obeys GAPP, there is an augmented utility function U such that

$$x^t \in \operatorname*{argmax}_{x \in \mathbb{R}^L_+} U(q^t, -q^t x^t + \delta^t) \text{ for all } t \in T, \tag{10}$$

and if $\mathcal{D}$ violates GAPP, then there is no augmented utility function U such that (10) holds.

Proof. Our proof of part (1) relies on a result of Varian (1988), which says that given any data set $\mathcal{D}$, there always exist $\{z^t\}_{t=1}^T$ such that the augemented data set $\{((p^t, 1), (x^t, z^t))\}_{t=1}^T$ obeys GARP. By Afriat's Theorem, there is a utility function $\widetilde{U} : \mathbb{R}^{L+1} \to \mathbb{R}$ such that (x^t, z^t) is optimal in the budget set $\{(x, z) \in \mathbb{R}^L \times \mathbb{R} \, : \, p^t x + z \leq M^t\}$, where $M^t = z^t + p^t x^t$. Afriat's Theorem also guarantees that this utility function has various nice properties and, in particular, $\widetilde{U}$ can be chosen to be strictly increasing. Let $\bar{M} = \sum_{t=1}^{T} M^t/T$ and define the augmented utility function $U : \mathbb{R}^L_+ \times \mathbb{R}_- \to \mathbb{R}$ by $U(x, -e) := \widetilde{U}(x, \bar{M} - e)$. Then since $\widetilde{U}$ is strictly increasing, x^t solves $\max_{x \in \mathbb{R}^L_+} U(x, \delta^t - p^t x)$ where $\delta^t = M^t - \bar{M}$.

There are two claims in (2). We first consider the case where $\mathcal{D}$ obeys GAPP. Note that whenever $p^{t'}x^{t'} - p^t x^{t'} < 0$, then for any $\{q^t\}_{t=1}^T$ and $\{\delta^t\}_{t=1}^T$ such that (9) holds, we obtain $p^{t'}x^{t'} - p^t x^{t'} < \delta^{t'} - \epsilon^{t'}x^{t'} - \delta^t + \epsilon^t x^{t'}$. This inequality can be re-written as

$$-q^t x^{t'} + \delta^t < -q^{t'} x^{t'} + \delta^{t'}. \tag{11}$$

Choose a number $M > \max_t(q^t x^t - \delta^t)$ and define the data set $\widetilde{\mathcal{D}} = \{(q^t, 1), (x^t, M + \delta^t - q^t x^t)\}_{t=1}^T$. Since (11) holds whenever $p^{t'}x^{t'} - p^t x^{t'} < 0$,

$$(q^t, 1)(x^t, M + \delta^t - q^t x^t) > (q^t, 1)(x^{t'}, M + \delta^{t'} - q^{t'} x^{t'}) \text{ only if } p^{t'}x^{t'} > p^t x^{t'}.$$

This guarantees that $\widetilde{\mathcal{D}}$ obeys GARP since $\mathcal{D}$ obeys GAPP. By Afriat's Theorem, there is a strictly increasing utility function $\widetilde{U} : \mathbb{R}^L \to \mathbb{R}$ such that $(x^t, M + \delta^t - q^t x^t)$ is optimal in the budget set $\{(x, z) \in \mathbb{R}^{L+1} : q^t x + z \leq M + \delta^t\}$. Define the augmented utility function $U : \mathbb{R}^L_+ \times \mathbb{R}_- \to \mathbb{R}$ by $U(x, -e) := \widetilde{U}(x, M - e)$. Since $\widetilde{U}$ is strictly increasing, x^t solves $\max_{x \in \mathbb{R}^L_+} U(x, \delta^t - q^t x)$.

Now consider the case where $\mathcal{D}$ violates GAPP. Observe that whenever $p^{t'}x^{t'} - p^t x^{t'} > 0$, then for any $\{q^t\}_{t=1}^T$ and $\{\delta^t\}_{t=1}^T$ such that (9) holds, we obtain $p^{t'}x^{t'} - p^t x^{t'} > \delta^{t'} - \epsilon^{t'}x^{t'} - \delta^t + \epsilon^t x^{t'}$. This inequality can be re-written as

$$-q^t x^{t'} + \delta^t > -q^{t'} x^{t'} + \delta^{t'}. \tag{12}$$

By way of contradiction, suppose there is an augmented utility function U such that x^t maximizes $U(x, -q^t x + \delta^t)$ for all t. For any observation t, we write $V^t := \max_{x \in \mathbb{R}^L} U(x, -q^t x + \delta^t)$. Since (12) holds, we obtain

$$V^t \geq U(x^{t'}, -q^t x^{t'} + \delta^t) > U(x^{t'}, -q^{t'} x^{t'} + \delta^{t'}) = V^{t'}.$$

Thus, we have shown that $V^t > V^{t'}$ whenever $p^{t'}x^{t'} - p^t x^{t'} > 0$. Since $\mathcal{D}$ violates GAPP there is a finite sequence $\{(p^{t_1}, x^{t_1}), \ldots, (p^{t_N}, x^{t_N})\}$ of distinct elements in $\mathcal{D}$, such that $p^{t_i}x^{t_{i+1}} < p^{t_{i+1}}x^{t_{i+1}}$ for all $i \in \{1, \ldots, N-1\}$ and $p^{t_N}x^{t_1} < p^{t_1}x^{t_1}$. By the observation we have just made, we obtain $V^{t_1} > V^{t_2} > ...V^{t_N} > V^{t_1}$, which is impossible. □

2.5. *Comparing GAPP and GARP*

Recall that Example 1 in Section 2.2 is an example of a data set that obeys GARP but fails GAPP. We now present an example of a data set that obeys GAPP but fails GARP.

Example 2. Consider the data set consisting of the following two choices:

$$p^t = (2, 1)\,,\, x^t = (2, 1) \;\text{ and }\; p^{t'} = (1, 4)\,,\, x^{t'} = (0, 2).$$

These choices are shown in Figure 2. This is a classic GARP violation as $p^t \cdot x^t = 5 > 2 = p^t \cdot x^{t'}$ ($x^t \succ_x x^{t'}$) and $p^{t'} \cdot x^{t'} = 8 > 6 = p^{t'} \cdot x^t$ ($x^{t'} \succ_x x^t$). In words, each bundle is strictly cheaper than the other at the budget set corresponding to the latter observation. However, these choices satisfy GAPP as $p^{t'} \cdot x^{t'} = 8 > 2 = p^t \cdot x^{t'}$ ($p^t \succ_p p^{t'}$) but $p^t \cdot x^t = 5 \not> 6 = p^{t'} \cdot x^t$ ($p^{t'} \not\succ_p p^t$).

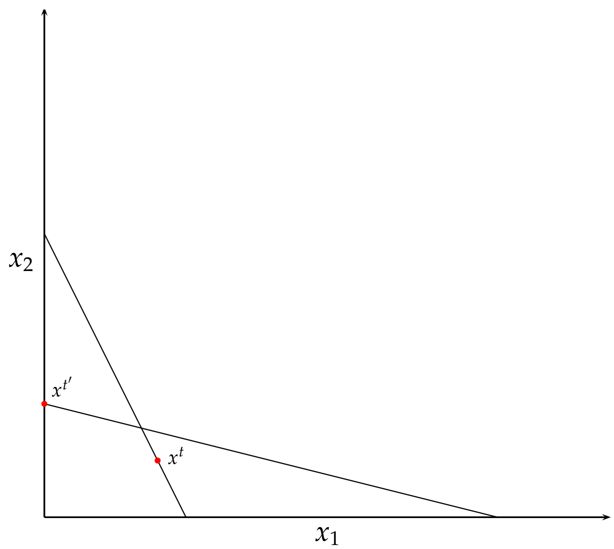


FIGURE 2. Choices that satisfy GAPP but not GARP

The upshot of this example is that there are data sets that admit rationalization with an augmented utility function that cannot be rationalized in Afriat's sense (that is, in the sense given by (1)). If we interpret the augmented utility function in the form $G(x, y)$ (given by (7)), then this means that while the agent's behavior is consistent with the maximization of an overall utility function G, this utility function is not weakly separable in the observed goods x. In particular, this implies that the agent does not have a quasilinear utility function (with $G(x, y) = U(x) + y$), which *is* weakly separable in the observed goods and will generate data sets obeying both GAPP and GARP.

While GAPP and GARP are not comparable conditions, there is a way of converting a GAPP test into a GARP test that will prove to be very convenient for us. Given a data set $\mathcal{D} = \{(p^t, x^t)\}_{t=1}^T$, we define the *expenditure-normalized* version of $\mathcal{D}$ as another data set $\widetilde{\mathcal{D}} := \left\{(p^t, \tilde{x}^t)\right\}_{t=1}^T$, such that $\tilde{x}^t = x^t / p^t x^t$. This new data set has the feature that $p^t \tilde{x}^t = 1$ for all $t \in T$. Notice that the revealed price preference relations $\succeq_p$, $\succ_p$ remain unchanged when consumption bundles are scaled. Put differently, a data set obeys GAPP if and only if its normalized version also obeys GAPP. Therefore, from the perspective of testing for rationalization (by an augmented utility function) this change in the data set is immaterial. The next proposition makes a different and less obvious observation about $\widetilde{\mathcal{D}}$.

Proposition 2. *Let $\mathcal{D} = \{(p^t, x^t)\}_{t=1}^T$ be a data set and let $\widetilde{\mathcal{D}} = \{(p^t, \tilde{x}^t)\}_{t \in T}$, where $\tilde{x}^t = x^t / (p^t x^t)$, be its expenditure-normalized version. Then the revealed preference relations $\succsim_p^*$ and $\succ_p^*$ on $\mathcal{P} = \{p^t\}_{t=1}^T$ and the revealed preference relations $\succsim_x^*$ and $\succ_x^*$ on $\widetilde{\mathcal{X}} = \{\tilde{x}^t\}_{t=1}^T$ are related in the following manner:*

(1) $p^t \succsim_p^* p^{t'}$ if and only if $\tilde{x}^t \succsim_x^* \tilde{x}^{t'}$.
(2) $p^t \succ_p^* p^{t'}$ if and only if $\tilde{x}^t \succ_x^* \tilde{x}^{t'}$.

As a consequence, $\mathcal{D}$ obeys GAPP if and only if $\widetilde{\mathcal{D}}$ obeys GARP.

Proof. Notice that

$$p^t \frac{x^t}{p^t x^t} \geq p^t \frac{x^{t'}}{p^{t'} x^{t'}} \iff p^{t'} x^{t'} \geq p^t x^{t'}.$$

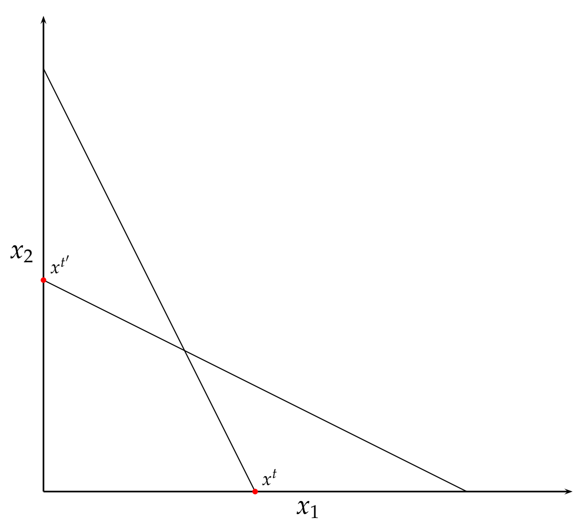
The left side of the equivalence says that $\tilde{x}^t \succsim_x \tilde{x}^{t'}$ while the right side says that $p^t \succsim_p p^{t'}$. This implies (1) since $\succsim_p^*$ and $\succsim_x^*$ are the transitive closures of $\succsim_p$ and $\succsim_x$ respectively. Similarly, it follows from

$$p^t \frac{x^t}{p^t x^t} > p^t \frac{x^{t'}}{p^{t'} x^{t'}} \iff p^{t'} x^{t'} > p^t x^{t'}$$

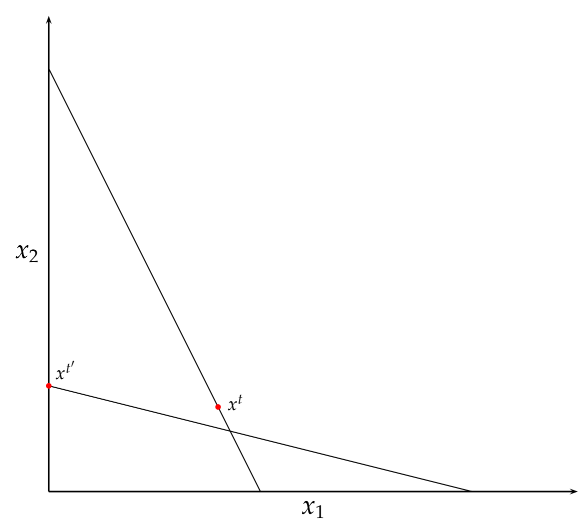
that $\tilde{x}^t \succ_x \tilde{x}^{t'}$ if and only if $p^t \succ_p p^{t'}$, which leads to (2). The claims (1) and (2) together guarantee that there is a sequence of observations in $\mathcal{D}$ that lead to a GAPP violation if and only if the analogous sequence in $\widetilde{\mathcal{D}}$ lead to a GARP violation. □

As an illustration of how Proposition 2 'works,' compare the data sets in Figure 1 and Figure 2 to the expenditure-normalized data sets in Figure 3a and Figure 3b. It can be clearly observed that the expenditure-normalized data in Figure 3a contains a GARP violation (which implies it does not satisfy GAPP) whereas the data in Figure 3b does not violate GARP (and, hence, satisfies GAPP).

A consequence of Proposition 2 is that the expenditure-augmented utility model can be tested in two ways: (i) we can test GAPP directly, or, (ii) we can normalize the data by expenditure and then test GARP. If we are simply interested in testing GAPP on a data set $\mathcal{D}$ then both methods are computationally straightforward and there is not much to choose between them: they both require the construction of their (respective) revealed preference relations and testing involves checking for acyclicity. However, as we shall see in the next section, the indirect procedure supplied by Proposition 2 will prove to be very useful for testing in the random augmented utility environment.



(a) Example 1



(b) Example 2

FIGURE 3. Expenditure-Normalized Choices

2.6. *Preference over Prices*

We know from Theorem 1 that if $\mathcal{D}$ obeys GAPP then it can be rationalized by an augmented utility function with an indirect utility that is defined at all price vectors in $\mathbb{R}^L_{++}$. It is straightforward to check that any indirect utility function V as defined by (5) has the following two properties:

(a) it is *nonincreasing in* p, in the sense that if $p' \geq p$ (in the product order) then $V(p') \leq V(p)$, and
(b) it is *quasiconvex in* p, in the sense that if $V(p) = V(p')$, then $V(\beta p + (1-\beta)p') \leq V(p)$ for any $\beta \in [0,1]$.

Any rationalizable data set $\mathcal{D}$ could potentially be rationalized by many augmented utility functions and each one of them will lead to a different indirect utility function. We denote this set of indirect utility functions by $\mathbf{V}(\mathcal{D})$. We have already observed that if $p^t \succeq_p^* (\succ_p^*) p^{t'}$ then $V(p^t) \geq (>) V(p^{t'})$ for any $V \in \mathbf{V}(\mathcal{D})$; in other words, the conclusion that the consumer prefers the prices p^t to $p^{t'}$ is *fully nonparametric* in the sense that it is independent of the precise augmented utility function used to rationalize $\mathcal{D}$. The next result says that, without further information on the agent's augmented utility function, this is *all* the information on the agent's preference over prices in $\mathcal{P}$ that we could glean from the data. Thus, in our nonparametric setting, the revealed price preference relation contains the most detailed information for welfare comparisons.

Proposition 3. *Suppose $\mathcal{D} = \{(p^t, x^t)\}_{t=1}^T$ is rationalizable by an augmented utility function. Then for any p^t, $p^{t'}$ in $\mathcal{P}$:*

(1) $p^t \succeq_p^* p^{t'}$ if and only if $V(p^t) \geq V(p^{t'})$ for all $V \in \mathbf{V}(\mathcal{D})$.
(2) $p^t \succ_p^* p^{t'}$ if and only if $V(p^t) > V(p^{t'})$ for all $V \in \mathbf{V}(\mathcal{D})$.

Proof. (1) We have already shown the 'only if' part of this claim, so we need to show the 'if' part holds. From the proof of Theorem 1, we know that for a large M, it is the case that $p^t \succeq_p p^{t'}$ if and only if $(x^t, M - p^t x^t) \succeq_x (x^{t'}, M - p^{t'} x^{t'})$ and hence $p^t \succeq_p^* p^{t'}$ if and only if $(x^t, M - p^t x^t) \succeq_x^* (x^{t'}, M - p^{t'} x^{t'})$. If $p^t \not\succeq_p^* p^{t'}$, then $(x^t, M - p^t x^t) \not\succeq_x^* (x^{t'}, M - p^{t'} x^{t'})$ and hence there is a utility function $\widetilde{U} : \mathbb{R}^{L+1}_+ \to \mathbb{R}$ rationalizing the augmented data set $\widetilde{D}$ such that $\widetilde{U}(x^t, M - p^t x^t) < \widetilde{U}(x^{t'}, M - p^{t'} x^{t'})$ (see Remark 2 in Section 2.1). This in turn implies that the augmented utility function U (as defined by (6)), has the property that $U(x^t, -p^t x^t) < U(x^{t'}, -p^{t'} x^{t'})$ or, equivalently, $V(p^t) < V(p^{t'})$.

(2) Given part (1), we need only show that if $p^t \succeq_p^* p^{t'}$ but $p^t \not\succ_p^* p^{t'}$, then there is some augmented utility function U such that $U(x^t, -p^t x^t) = U(x^{t'}, -p^{t'} x^{t'})$. To see that this holds, note that if $p^t \succeq_p^* p^{t'}$ but $p^t \not\succ_p^* p^{t'}$, then $(x^t, M - p^t x^t) \succeq_x^* (x^{t'}, M - p^{t'} x^{t'})$ but $(x^t, M - p^t x^t) \not\succ_x^* (x^{t'}, M - p^{t'} x^{t'})$. In this case there is a utility function $\widetilde{U} : \mathbb{R}^{L+1}_+ \to \mathbb{R}$ rationalizing the augmented data set $\widetilde{D}$ such that $\widetilde{U}(x^t, M - p^t x^t) = \widetilde{U}(x^{t'}, M - p^{t'} x^{t'})$. This in turn implies that the augmented utility function U (as defined by (6)) satisfies $U(x^t, -p^t x^t) = U(x^{t'}, -p^{t'} x^{t'})$ and so $V(p^t) = V(p^{t'})$. □

3. The Stochastic Model

In this section, we develop the stochastic version of expenditure-augmented utility model. Following the treatment we adopted in the deterministic case, we shall begin with an explanation of the standard version of the random utility model, as found in McFadden and Richter (1991) and Kitamura and Stoye (2013) (henceforth to be referred to, respectively, as MR and KS).

3.1. *Rationalization by Random Utility*

Suppose that instead of observing single choices on T budget sets, the econometrician observes choice probabilities on each budget set. Our preferred interpretation is that each observation corresponds to the distribution of choices made by a population of consumers and the data set consists of a repeated cross section of such choice probabilities. This interpretation is appropriate for our empirical application but an alternative interpretation is that the econometrician observes data from a single individual who makes multiple choices at each budget set.

We denote the budget set corresponding to observation t by $B^t := \{x \in \mathbb{R}^L_+ \ : \ p^t x = 1\}$. In this model, only relative prices matter, so we can scale prices and normalize income to 1 without loss of generality. We use $\breve{\pi}^t$ to denote the probability measure of choices on budget set B^t at observation t. Thus, for any subset $X^t \subset B^t$, $\breve{\pi}^t(X^t)$ denotes the probability that the choices lie in the subset X^t. Following MR and KS, we assume that the econometrician observes the stochastic data set $\mathscr{D} := \{(B^t, \breve{\pi}^t)\}_{t=1}^T$, which consists of a finite collection of budget sets along with the corresponding choice probabilities.

For ease of exposition, we also impose the following assumption on the data:

$$\text{for all } t, t' \in T \text{ with } B^t \neq B^{t'}, \text{ the choice probabilities satisfy } \breve{\pi}^t\left(\{B^t \cap B^{t'}\}\right) = 0. \tag{A1}$$

In other words, the choice probability measure $\breve{\pi}$ has no mass where the budget sets intersect. This is convenient because it simplifies some of the definitions that follow.[13]

A random utility is denoted by a measure $\widetilde{\mu}$ over the set of locally nonsatiated utility functions defined on $\mathbb{R}^L_+$, which we denote by $\widetilde{\mathscr{U}}$. The data set $\mathscr{D}$ is said to be *rationalized by a random utility model* if there exists a random utility $\tilde{\mu}$ such that for all $X_t \subset B^t$,

$$\breve{\pi}^t(X^t) = \widetilde{\mu}(\widetilde{\mathscr{U}}(X^t)) \ \text{ for all } \ t \in T, \ \text{ where } \ \widetilde{\mathscr{U}}(X^t) := \left\{ \widetilde{U} \in \widetilde{\mathscr{U}} \ : \ \operatorname*{argmax}_{x \in B^t} \widetilde{U}(x) \in X^t \right\}.$$

In other words, to rationalize $\mathscr{D}$ we need to find a distribution on the family of utility functions that generates a demand distribution at each budget set B^t corresponding to what was observed. The crucial fact that this problem can be solved via a finite procedure was noted by MR. We now explain the method they proposed.

Let $\{B^{1,t}, \ldots, B^{I_t,t}\}$ denote the collection of subsets (called *patches* in KS) of the budget B^t where each subset has as its boundaries the intersection of B^t with other budget sets and/or the boundary hyperplanes of the positive orthant. Formally, for all $t \in T$ and $i_t \neq i'_t$, each set in $\{B^{1,t}, \ldots, B^{I_t,t}\}$ is closed and convex and, in addition, the following hold:

[13]This simplification is not conceptually necessary for the procedure described here or its adaptation to our setting in the next subsection. See the explanations given in KS, all of which also apply here.

(i) $\cup_{1\leq i_t\leq I_t} B^{i_t,t} = B^t$,
(ii) $\text{int}(B^{i_t,t}) \cap B^{t'} = \phi$ for all $t' \neq t$ that satisfy $B^t \neq B^{t'}$,
(iii) $B^{i_t,t} \cap B^{i'_t,t} \neq \phi$ implies that $B^{i_t,t} \cap B^{i'_t,t} \subset B^{t'}$ for some $t' \neq t$ that satisfies $B^t \neq B^{t'}$,

where $\text{int}(\cdot)$ denotes the interior of a set.

We use the vector $\pi^t \in \Delta^{I_t}$ belonging to the I^t dimensional simplex to denote the *discretized choice probabilities* over the collection $\{B^{1,t}, \ldots, B^{I_t,t}\}$. Formally, coordinate i_t of π^t is given by

$$\pi^{i_t,t} = \breve{\pi}^t \left(B^{i_t,t}\right), \quad \text{for all } B^{i_t,t} \in \left\{B^{1,t}, \ldots, B^{I_t,t}\right\}.$$

Even though there may be t, i_t, i'_t for which $B^{i_t,t} \cap B^{i'_t,t} \neq \phi$ (as these sets may share parts of their boundaries), Assumption A1 guarantees that π^t is still a probability measure since choice probabilities on the boundaries of $B^{i_t,t}$ have measure 0. We denote $\pi := (\pi^1, \ldots, \pi^T)'$.

We call a deterministic data set $\mathcal{D} = \{(p^t, x^t)\}_{t=1}^T$ *typical* if, for all t, there is i^t such that $x^t \in \text{int}(B^{i^t,t})$ for all $t \in T$; in other words, x^t lies in the interior of some patch at each observation. If a typical deterministic data set $\mathcal{D} = \{(p^t, x^t)\}_{t=1}^T$ satisfies GARP, then for all other $\breve{x}^t \in \text{int}(B^{i^t,t})$, $t \in T$, the data set $\{(p^t, \breve{x}^t)\}_{t=1}^T$ also satisfies GARP. This is because the revealed preference relations $\succeq_x$, $\succ_x$ are determined only by where a choice lies on the budget set relative to its intersection with another budget. Thus, as far as testing rationality is concerned, all choices in a given set $\text{int}(B^{i^t,t})$ are interchangeable. Therefore, we may classify all typical deterministic data sets according to the patch occupied by the bundle x^t at each budget set B^t. In formal terms, we associate to each typical deterministic data set $\mathcal{D}$ that satisfies GARP a vector $a = \left(a^{1,1}, \ldots, a^{I_T,T}\right)$ where

$$a^{i_t,t} = \begin{cases} 1 & \text{if } x^t \in B^{i_t,t}, \\ 0 & \text{otherwise.} \end{cases} \tag{13}$$

Notice that we have now partitioned all typical deterministic data sets obeying GARP (of which there are infinitely many) into a *finite* number of distinct classes or types, based on the vector a associated with each data set. We use A to denote the matrix whose columns consist of all the a vectors corresponding to these GARP-obeying types (where the columns can be arranged in any order) and use $|A|$ to denote the number of such columns (which is also the number of types).

The problem of finding a measure $\tilde{\mu}$ on the family of utility functons to rationalize $\mathscr{D}$ is essentially one of disaggregating $\mathscr{D}$ into rationalizable deterministic data sets or, given Afriat's Theorem, into deterministic data sets obeying GARP. If we ignore non-typical deterministic data sets (which is justified because of Assumption A1), this is in turn equivalent to finding weights on the finitely many distinct types represented by the columns of A, so that their aggregation at each observation coincides with the discretized choice probabilities. The following result summarizes these observations.

MR Theorem. *Suppose that the stochastic data set $\mathscr{D} = \{(B^t, \breve{\pi}^t)\}_{t=1}^T$ satisfies Assumption A1. Then $\mathscr{D}$ is rationalized by a random utility model if and only if there exists a $\nu \in \mathbb{R}_+^{|A|}$ such that the discretized choice probabilities π satisfy $A\nu = \pi$.*[14]

[14]Since any ν that satisfies $A\nu = \pi$ must also satisfy $\sum_{j=1}^{|A|} \nu_j = 1$, we do not need to explicitly impose the latter condition.

Before we turn to the stochastic version of the expenditure-augmented utility model, it is worth highlighting an important shortcoming of the setting envisaged by MR : data sets of the form $\mathscr{D} = \{(B^t, \breve{\pi}^t)\}_{t=1}^T$ are typically unavailable. This is because even consumers that face the same prices on the L observed goods (as is commonly assumed in applications with repeated cross sectional data, for instance, Blundell, Browning, and Crawford (2008)) will typically spend different amounts on these goods and, therefore, the observed choices of the population will lie on different budget sets; furthermore, even if one conditions on that part of the population that chooses the same expenditure level at some vector of prices, as these prices change, they will go on to choose different expenditure levels. The procedure set out by MR to test the random utility model relies heavily on the stylized environment they assume and it is not at all clear how it could be modified to test the model when expenditure is allowed to be endogenous. Recently, the procedure they proposed was implemented by KS but, in order to do this, KS first had to (in effect) *create* a data set of the MR type: this involves estimating (from the actual data) what the distribution of demand would be if, hypothetically, all consumers were restricted to the same budget set. Inevitably, estimating these distributions will require the use of instrumental variables (to get round the endogeneity of expenditure) and involve a variety of assumptions on the smoothness of Engel curves, the nature of unobserved heterogeneity across individuals, etc.

3.2. *Rationalization by Random Augmented Utility*

Once again the starting point of our analysis is a stochastic data set $\mathscr{D} := \{(p^t, \tilde{\pi}^t)\}_{t=1}^T$ which consists of a finite set of distinct prices along with a corresponding distribution over chosen bundles. But there is one important departure from the previous section: we no longer require the support of $\tilde{\pi}^t$ to lie on the budget set B^t; instead the support could be any set in $\mathbb{R}_+^L$. In other words, we no longer require all consumers to incur the same expenditure at each price observation; each consumer in the population can decide how much she wishes to spend on the L observed goods and this could differ across consumers and across price observations. As we pointed out at the end of Section 3.2, this is the form that data typically takes.

A random expenditure-augmented utility is denoted by a measure μ over the set of augmented utility functions which we denote by $\mathscr{U}$.

Definition 3.1. The data set $\mathscr{D}$ is said to be **rationalized by the random augmented utility model** if there exists a random augmented utility μ such that for all $X_t \subset \mathbb{R}_+^L$,

$$\tilde{\pi}^t(X^t) = \mu(\mathscr{U}(X^t)) \text{ for all } t \in T, \text{ where } \mathscr{U}(X^t) := \left\{ U \in \mathscr{U} \ : \operatorname*{argmax}_{x \in \mathbb{R}_+^L} U(x, -p^t x) \in X^t \right\}.$$

In actual empirical applications, observations are typically made over time. Therefore, we are effectively asking whether or not $\mathscr{D}$ is generated by a distribution of augmented utility functions that is stable over the period where observations are taken. This assumption is plausible if (i) there is no change in the prices of the unobserved goods or, more realistically, that these changes could be adequately accounted for by the use of a deflator, and (ii) there is sufficient stability in the way consumers in the population view their long term economic prospects, so that there is no change in their habitualized willingness to trade off consumption of a bundle of goods in L with the expenditure it incurs.

The problem of finding a measure $\tilde{\mu}$ to rationalize $\mathscr{D}$ is essentially one of disaggregating $\mathscr{D}$ into deterministic data sets rationalizable by augmented utility functions or, given Theorem 1, into deterministic data sets obeying GAPP. Crucially, Proposition 2 tells us that a deterministic data set obeys GAPP if and only if its expenditure-normalized version obeys GARP. It follows that $\tilde{\mu}$ exists if and only if the normalized version of the stochastic data set $\mathscr{D}$ obeys the condition identified by MR (as stated in Section 3.2).

To set this out more formally, we first define the *normalized choice probability* $\breve{\pi}$ corresponding to $\tilde{\pi}$ by scaling observations from the entire orthant onto the budget plane B^t generated by prices p^t and expenditure 1. Formally,

$$\breve{\pi}^t(X^t) = \tilde{\pi}^t\left(\left\{x : \frac{x}{p^t x} \in X^t\right\}\right), \quad \text{for all } X^t \subset B^t \text{ and all } t \in T.$$

We suppose that Assumption A1 holds on the normalized data set $\{(B^t, \breve{\pi}^t)\}_{t=1}^T$; abusing the terminology somewhat, we shall say that $\mathscr{D}$ obeys Assumption A1.[15] We then define the patches on the budgets set B^t (as in Section 3.2) and denote them by $\{B^{1,t}, \ldots, B^{I_t,t}\}$. With these patches in place, we derive the *normalized and discretized choice probabilities* $\pi^t = \{\pi^{1,t}, \ldots, \pi^{I_t,t}\}$ from $\breve{\pi}^t$ by assigning to each $\pi^{i_t,t}$ the normalized choice probability $\breve{\pi}^t(B^{i_t,t})$ corresponding to $B^{i_t,t}$. Finally, we construct the matrix A, whose columns are defined by (13); the columns represent distinct GARP-obeying types, which by Proposition 2 coincides with the distinct GAPP-obeying types. The rationalizability of $\mathscr{D}$ can then be established by checking if there are weights on these types that, at each observation, generate the observed normalized choice probabilities. The following result summarizes these observations.

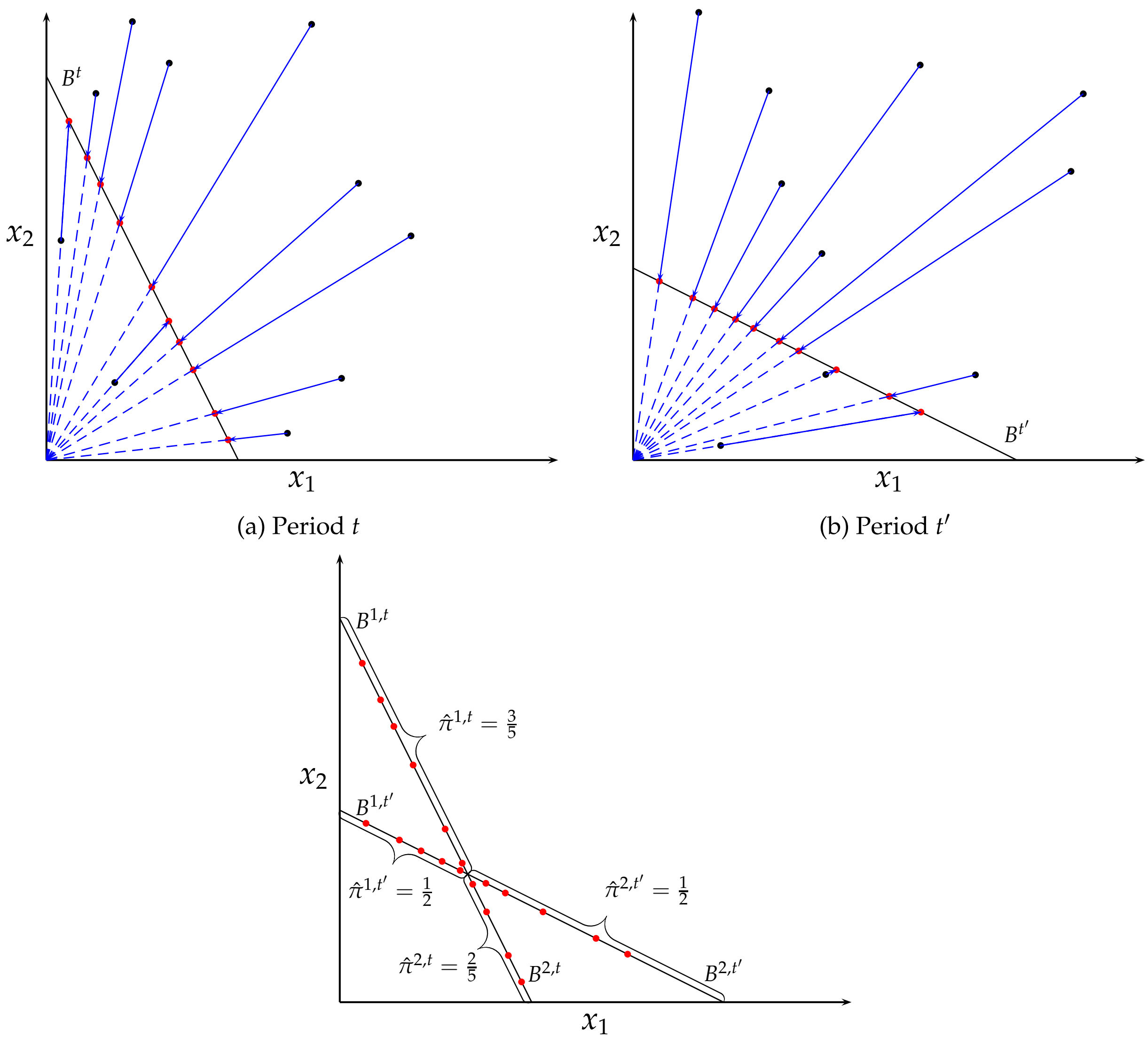
Theorem 2. *Let $\mathscr{D} = \{(p^t, \tilde{\pi}^t)\}_{t=1}^T$ be a stochastic data set obeying Assumption A1. Then $\mathscr{D}$ is rationalized by the random augmented utility model if and only if there exists a $\nu \in \mathbb{R}_+^{|A|}$ such that the normalized and discretized choice probabilities π satisfy $A\nu = \pi$.*

It is worth emphasizing that this theorem provides us with a very clean procedure for testing the random augmented utility model. If we were testing the random utility model, then the MR test requires a data set where expenditures are common across consumers as a starting point; since this is not commonly available it would have to be estimated, which in turn requires an additional econometric procedure with all its attendant assumptions. By contrast, to test the random augmented utility model, all we have to do is apply the MR test to the expenditure-normalized data set $\{(B^t, \pi^t)\}_{t=1}^T$, which is obtained (via a simple transformation) from the original data set $\mathscr{D} = \{(p^t, \tilde{\pi}^t)\}_{t=1}^T$. This allows us to determine the rationalizability of $\mathscr{D}$ because we know, purely as a consequence of the theoretical model itself and without any further assumptions, that one data set is rationalized by a random augmented utility model if and only if the other one is.

[15] A sufficient condition for A1 is that $\tilde{\pi}$ assigns 0 probability to sets with a Lebesgue measure of 0. Also, as before, this is merely for ease of exposition: our test does not depend on this assumption and, importantly, the data in our empirical application satisfies A1.

We end this subsection with an example that makes explicit the operationalization of Theorem 2 using data where the normalized and discretized choice probabilities are determined by the sample frequency of choices.

Example 3. Suppose the econometrician observes the set of ten choices at two price vectors, $p^t = (2,1)$ and $p^{t'} = (1,2)$, given by the black points in Figures 4a and 4b. These figures also



(a) Period t

(b) Period t'

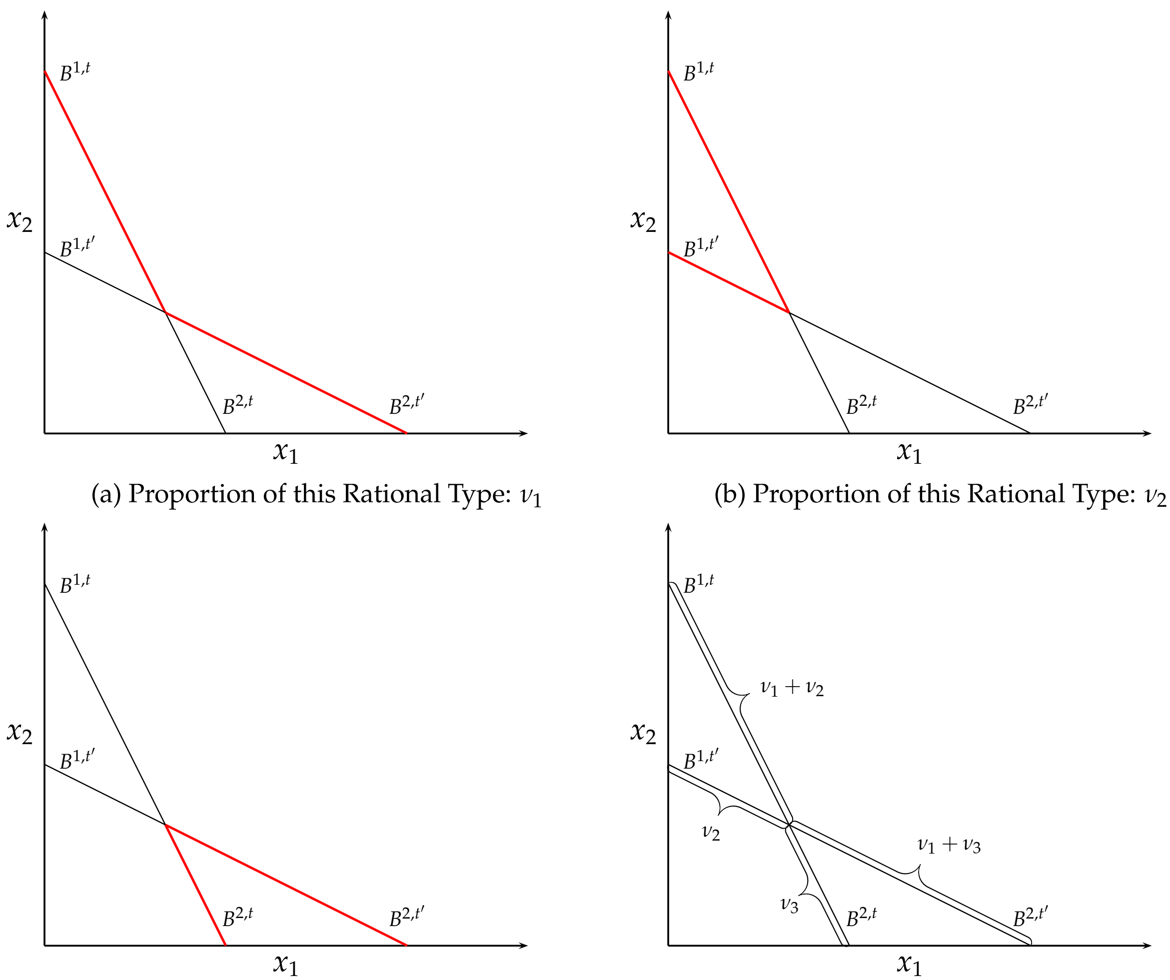
(c) The Normalized and Discretized Empirical Choice Probabilities $\hat{\pi}$

FIGURE 4. Observed and Scaled Choice Data

demonstrate how the choices are scaled (to the red points) on to the normalized budget sets. Figure 4c then shows that the choice probabilities

$$\hat{\pi} = \left(\frac{3}{5}, \frac{2}{5}, \frac{1}{2}, \frac{1}{2}\right)'$$

(the hat notation refers to the fact that the choice probabilities are derived from the sample choice frequencies which is how we estimate choice probabilities) are determined by the proportion of the normalized choices that lie on each segment of the budget lines. Lastly, Figure 5 illustrates the various rational types for these two budget sets. Note that GARP (equivalently, GAPP) violations



(a) Proportion of this Rational Type: ν_1

(b) Proportion of this Rational Type: ν_2

(c) Proportion of this Rational Type: ν_3

(d) Resulting Distribution of Choices

FIGURE 5. Distribution of Rational Types

only occur when the choices lie on $B^{2,t}$ and $B^{1,t'}$. The resulting A matrix is given by

$$A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \text{ and, additionally, } A\nu = \begin{pmatrix} \nu_1 + \nu_2 \\ \nu_3 \\ \nu_2 \\ \nu_1 + \nu_3 \end{pmatrix}.$$

The first, second and third column of A correspond to the three types of GARP-consistent demand behavior, which are depicted in Figures 5a, 5b and 5c respectively. If the proportion of the three types in the population is ν_1, ν_2 and ν_3, the resulting distribution on the segments of the budget sets are given by $A\nu$, the expression for which is displayed above and depicted in Figure 5d. Theorem 2 says that rationalization is equivalent to the existence of $\nu \in \mathbb{R}^3_+$ such that $A\nu = \hat{\pi}$.

The data from this example can be rationalized by the distribution of rational types

$$\nu = \left(\frac{1}{10}, \frac{1}{2}, \frac{2}{5}\right)'.$$

Notice also that it is not the case that a solution always exists. Indeed, if $\hat{\pi}^{1,t'} > \hat{\pi}^{1,t}$, then the choice probabilities would not be rationalizable as $\nu_2 > \nu_1 + \nu_2$ is not possibile.

3.3. *Welfare Comparisons*

Since the test for rationalizability involves finding a distribution ν over different types, it is possible to use this distribution for welfare comparisons: for any two prices in the data set and given a distribution ν that rationalizes $\mathscr{D}$, we can determine the proportion of types who are revealed better off and the proportion who are revealed worse off. However, since there may be multiple ν that satisfy $A\nu = \pi$, the welfare rankings extractable from the data will typically be in terms of bounds.

To be specific, suppose we would like to determine the welfare effect of a price change from $p^{t'}$ to p^t. Let $\mathbb{1}_{t \succeq^*_p t'}$ denote a vector of length $|A|$ such that the j^{th} element is 1 if $p^t \succeq^*_p p^{t'}$ for the rational type corresponding to column j of A and 0 otherwise. In words, $\mathbb{1}_{t \succeq^*_p t'}$ enumerates the set of rational types for which p^t is revealed preferred to $p^{t'}$. For a rationalizable data set $\mathscr{D}$, the solution to the optimization problem

$$\underline{\mathcal{N}}_{t \succeq^*_p t'} := \begin{array}{c} \min_\nu \ \mathbb{1}_{t \succeq^*_p t'} \, \nu, \\ \text{subject to } \ A\nu = \pi, \end{array} \tag{14}$$

gives the lower bound on the proportion of consumers who are revealed better off at p^t compared to $p^{t'}$, while

$$\overline{\mathcal{N}}_{t \succeq^*_p t'} := \begin{array}{c} \max_\nu \ \mathbb{1}_{t \succeq^*_p t'} \, \nu, \\ \text{subject to } \ A\nu = \pi, \end{array} \tag{15}$$

gives the upper bound on the proportion of consumers who are revealed better off at prices p^t compared to $p^{t'}$. In a similar way, we can find $\underline{\mathcal{N}}_{t' \succeq^*_p t}$ and $\overline{\mathcal{N}}_{t' \succeq^*_p t}$, the lower and upper bounds on the proportion of consumers who are revealed better off at $p^{t'}$ compared to p^t.

Since (14) and (15) are both linear programming problems (which are guaranteed to have solutions when $\mathscr{D}$ is rationalizable), they are easy to implement and computationally efficient. Suppose that the solutions are $\underline{\nu}$ and $\bar{\nu}$ respectively; then for any $\beta \in [0,1]$, $\beta\bar{\nu} + (1-\beta)\underline{\nu}$ is also a solution to $A\nu = \pi$ and, in this case, the proportion of consumers who are revealed better off at p^t compared to $p^{t'}$ is exactly $\beta \underline{\mathcal{N}}_{t \succeq^*_p t'} + (1-\beta)\overline{\mathcal{N}}_{t \succeq^*_p t'}$. In other words, the proportion of consumers who are revealed better off can take any value in the interval $[\underline{\mathcal{N}}_{t \succeq^*_p t'}, \overline{\mathcal{N}}_{t \succeq^*_p t'}]$.

Proposition 3 tells us that the revealed preference relations are tight, in the sense that if, for some consumer, p^t is not revealed preferred to $p^{t'}$ then there exists an augmented utility function which rationalizes her consumption choices and for which she strictly prefers $p^{t'}$ to p^t. Given this, we know that, amongst all rationalizations of $\mathscr{D}$, $\underline{\mathcal{N}}_{t \succeq^*_p t'}$ is also the infimum on the proportion of consumers who are better off at p^t compared to $p^{t'}$. At the other extreme, we know that there is a rationalization for which the proportion of consumers who are revealed better off at $p^{t'}$ compared to p^t is as low as $\underline{\mathcal{N}}_{t' \succeq_p^* t}$.[16] Applying Proposition 3 again, a rationalization could be chosen such that all other consumers prefer p^t to $p^{t'}$. Therefore, across all rationalizations of $\mathscr{D}$, $1 - \underline{\mathcal{N}}_{t' \succeq_p^* t}$ is the supremum on the proportion of consumers who prefer p^t and $p^{t'}$.

The following proposition summarizes these observations.

Proposition 4. *Let $\mathscr{D} = \{(p^t, \tilde{\pi}^t)\}_{t=1}^T$ be a stochastic data set that satisfies Assumption A1 and is rationalized by the augmented utility model.*

(i) *Then for every $\eta \in [\underline{\mathcal{N}}_{t \succeq_p^* t'}, \overline{\mathcal{N}}_{t \succeq_p^* t'}]$, there is rationalization of $\mathscr{D}$ for which η is the proportion of consumers who are revealed better off at p^t compared to $p^{t'}$.*

(ii) *For any rationalization of $\mathscr{D}$, there is a proportion of consumers who are better off at p^t compared to $p^{t'}$; let M be the set containing these proportions. Then* $\inf M = \underline{\mathcal{N}}_{t \succeq_p^* t'}$ *and* $\sup M = 1 - \underline{\mathcal{N}}_{t' \succeq_p^* t}$.

It may be helpful to consider how Proposition 4 applies to Example 3. In that case, there is a unique solution to $A\nu = \pi$ and the proportion of consumers who are revealed better off at p^t compared to $p^{t'}$ is $\nu_2 = 1/2$, while the proportion who are revealed better off at $p^{t'}$ to p^t is $\nu_3 = 2/5$. Those consumers who belong to neither of these two types could be either better or worse at p^t compared to $p^{t'}$. Therefore, across all rationalizations of that data set, the proportion of consumers who are better off at p^t compared to $p^{t'}$ can be as low as $1/2$ and as high as $1 - 2/5 = 3/5$.

4. The Econometric Methodology

We now turn to our econometric methodology. This will work through Theorem 2 and Proposition 4, which exhibit a tight link between the stochastic extension of our model and conventional random utility models. We answer two questions: First, how to statistically test GAPP from repeated cross-section data; second, how to conduct inference about the GAPP-constrained welfare bounds derived in Section 3.3. Our answer to the first question closely builds on the nonparametric test of Random Utility Models devised by KS. As we shall see, their approach naturally extends to GAPP; in fact, the present application will in some sense turn out simpler and maybe more natural than the original one. The second question calls for a novel approach, and we address it by proposing a method that is easy to implement in practice. The method potentially applies to inference about bounds for other parameters of interest as well, including in other settings that use the KS statistical test.

4.1. *Testing the Random Augmented Utility Model*

We test the Random Augmented Utility Model as outlined in Section 3.2 by invoking Theorem 2 and then drawing on KS. Since it wil be needed in the next section, we very briefly recapitulate their approach; additional relevant details (and limitations) are explained in Section 5. An equivalent and convenient way to state the hypothesis is

$$\min_{\nu \in \mathbb{R}_+^H} [\pi - A\nu]' \Omega [\pi - A\nu] = 0, \tag{16}$$

[16] Because of Assumption A1, we could assume that this is a strict revealed preference.

where Ω is a positive definite matrix and $H = |A|$. The solution of the above minimization problem is the projection of π onto the cone $\{A\nu : \nu \in \mathbb{R}^H_+\}$ under the weighted norm $\|x\|_\Omega = \sqrt{x'\Omega x}$. The corresponding value of the objective function is the squared length of the projection residual vector. Of course, choice probabilities π can be rationalized by a random augmented utility model if and only if the length of the residual vector is zero.

KS construct the following test statistic given by the natural sample counterpart of the objective function (16):

$$J_N := N \min_{\nu \in \mathbb{R}^H_+} [\hat{\pi} - A\nu]'\Omega[\hat{\pi} - A\nu], \tag{17}$$

where $\hat{\pi}$ estimates the normalized and discretized choice probabilities π by the sample choice frequencies (as in Figure 4c)[17] and N denotes the total number of observations (the sum of the number of households across years). The normalization by N is done in order to obtain an appropriate asymptotic distribution.

As we mentioned in the previous section, the minimizing value $\hat{\nu}$ may not be unique but the resulting choice probabilities $A\hat{\nu}$ are unique at the optimum. This latter term can be interpreted as a rationality-constrained estimator of choice probabilities. Note that $A\hat{\nu} = \hat{\pi}$ and $J_N = 0$, iff the sample choice frequencies can be rationalized by a random augmented utility model. In this case, the null hypothesis is trivially accepted. KS devise a procedure to estimate the distribution of J_N in order to determine the appropriate critical value for the test statistic.

4.2. *Estimating Welfare Bounds*

We can test whether a particular number $\mathcal{N}_{t \succeq^*_p t'}$ is in the welfare bounds from Proposition 4 by adding a linear constraint to the hypothesis (16). Formally, we test

$$\min_{\nu \in \mathbb{R}^H_+, \mathbb{1}_{t \succeq^*_p t'} \nu = \mathcal{N}_{t \succeq^*_p t'}} [\pi - A\nu]'\Omega[\pi - A\nu] = 0 \tag{18}$$

or equivalently, whether π is contained in the set

$$S(\mathcal{N}_{t \succeq^*_p t'}) = \{\bar{\pi} = A\nu | \mathbb{1}'_{t \succeq^*_p t'} \nu = \mathcal{N}_{t \succeq^*_p t'}, \nu \in \Delta^{H-1}\},$$

where Δ^{H-1} is the $H-1$-unit simplex; thus, the set collects all rationalizable vectors $\bar{\pi}$ compatible with the proportion $\mathcal{N}_{t \succeq^*_p t'}$.

A confidence interval is generated by inverting the hypothesis test. The challenge is to compute an appropriate critical value for this test statistic even though its limiting distribution discontinuously depends on nuisance parameters in a very complex manner.

Once again, the hypothesis being tested here takes the form

$$\pi \in S(\mathcal{N}_{t \succeq^*_p t'}). \tag{19}$$

Later in this section, we prove that this is equivalent to

$$\tilde{D}(\mathcal{N}_{t \succeq^*_p t'})\pi \leq 0, \tag{20}$$

[17] In the data we employ for our empirical application, there are no observations that lie on the intersection of any two normalized budget sets so $\hat{\pi}^t$ determines a probability measure on $\{B^{1,t}, \ldots, B^{I_t,t}\}$.

where $\tilde{D}$ is a matrix that depends on $\mathcal{N}_{t\succeq_p^* t'}$: in other words the matrix $\tilde{D}$, up to its equivalence class, is completely determined once $\mathcal{N}_{t\succeq_p^* t'}$ and prices $\{p_t\}_{t=1}^T$ are given. Notice that such a matrix $D(\mathcal{N}_{t\succeq_p^* t'})$ is guaranteed to exist *theoretically*, but calculating it for a problem of an empirically relevant scale is practically impossible. If it could be computed, then testing (20) for each value of $\mathcal{N}_{t\succeq_p^* t'}$, and obtaining a confidence interval for it reduces to the canonical moment inequality testing problem considered in Andrews and Soares (2010), since (20) is a system of moment inequalities indexed only by the value of $\mathcal{N}_{t\succeq_p^* t'}$ (which plays the role of the parameter "θ" that indexes the moment function in Andrews and Soares (2010)). Unfortunately, when $\tilde{D}(\mathcal{N}_{t\succeq_p^* t'})$ cannot be computed, the (now standard) Generalized Moment Selection approach of Andrews and Soares (2010) is difficult to implement since it requires numerical evaluation of the moment function, which cannot be done without knowing $D(\mathcal{N}_{t\succeq_p^* t'})$.[18]

Instead, we propose a practical method for testing and confidence interval calculation for $\mathcal{N}_{t\succeq_p^* t'}$ which sidesteps the calculation of $D(\mathcal{N}_{t\succeq_p^* t'})$ entirely. Our methodology builds upon the "tightening" approach in KS though our problem has features that are absent in the stochastic rationality hypothesis they consider. In particular, the correctness of the tightening method depends on the geometric properties of the set $S(\mathcal{N}_{t\succeq_p^* t'})$ which is more complicated in our setting than what is tested in KS. Moreover, the hypothesis considered in KS is not indexed by a parameter, whereas our hypothesis (19) depends on $\mathcal{N}_{t\succeq_p^* t'}$. Therefore the tightening approach needs to be tailored for this feature so that, in particular, it applies to cases where $\mathcal{N}_{t\succeq_p^* t'}$ is close to the boundary of its parameter space. This section address these problems, first verifying that $S(\mathcal{N}_{t\succeq_p^* t'})$ has the necessary geometric properties, then solving the second problem by proposing a method we call "restriction-dependent tightening" and demonstrating that it provides asymptotically valid tests and confidence intervals.

Define

$$\delta(\mathcal{N}_{t\succeq_p^* t'}) = \min_{\eta \in S(\mathcal{N}_{t\succeq_p^* t'})} [\pi - \eta]'\Omega[\pi - \eta].$$

Our test statistic replaces the unknown population value of π with the sample choice frequency $\hat{\pi}$ and scales the object:

$$\delta_N(\mathcal{N}_{t\succeq_p^* t'}) = \min_{\eta \in S(\mathcal{N}_{t\succeq_p^* t'})} N[\hat{\pi} - \eta]'\Omega[\hat{\pi} - \eta]$$

Recalling that $\mathbb{1}_{t\succeq_p^* t'}$ is binary, it is without further loss of generality to suppose that its first $\overline{H}$ elements of are ones and the remaining $H - \overline{H}$ elements are zeros. Then, writing $\nu = [\nu_1, ..., \nu_H]'$,

[18] An alternative method would be subsampling. While asymptotically valid, there is a concern about its power, as forcefully argued in Andrews and Soares (2010), page 137. The method we propose is theoretically much closer to Andrews and Soares (2010).

and $A = [\overline{A} \vdots \underline{A}]$, $\overline{A} = [a_1, ..., a_{\overline{H}}]$, $\underline{A} = [a_{\overline{H}+1}, ..., a_H]$

$$\begin{aligned} S(\mathcal{N}_{t \succeq_p^* t'}) &= \{\pi = A\nu | \sum_{j=1}^{\overline{H}} \nu_j = \mathcal{N}_{t \succeq_p^* t'}, \sum_{j=\overline{H}+1}^{H} \nu_j = 1 - \mathcal{N}_{t \succeq_p^* t'}, \nu \in \mathbb{R}_+^H\} \\ &= \{\pi = \mathcal{N}_{t \succeq_p^* t'} \overline{\pi} + (1 - \mathcal{N}_{t \succeq_p^* t'}) \underline{\pi} | \overline{\pi} = \sum_{j=1}^{\overline{H}} \overline{\nu}_j a_j, \overline{\nu}_j \geq 0, \\ &\qquad \sum_{j=1}^{\overline{H}} \overline{\nu}_j = 1, \underline{\pi} = \sum_{j'=1}^{H-\overline{H}} \underline{\nu_j} a_{j'+\overline{H}}, \underline{\nu_{j'}} \geq 0, \sum_{j'=1}^{H} \underline{\nu_{j'}} = 1\} \\ &= \mathcal{N}_{t \succeq_p^* t'} \mathrm{conv} \overline{A} \oplus (1 - \mathcal{N}_{t \succeq_p^* t'}) \mathrm{conv} \underline{A} \end{aligned}$$

where the $\oplus$ sign in the last line signifies Minkowski addition. By a basic property of Minkowski addition (e.g. Lemma 2 of Arrow and Hahn, 1971, p.387)

$$S(\mathcal{N}_{t \succeq_p^* t'}) = \mathrm{conv}\left(\{\mathcal{N}_{t \succeq_p^* t'} a_1, ..., \mathcal{N}_{t \succeq_p^* t'} a_{\overline{H}}\} \oplus \{(1 - \mathcal{N}_{t \succeq_p^* t'}))a_{\overline{H}+1}, ..., (1 - \mathcal{N}_{t \succeq_p^* t'}))a_{\overline{H}}\} \right).$$

Let $\{e_1, ..., e_K\}$, $K \leq \overline{H}(H - \overline{H})$, be the collection of elements in $\{\mathcal{N}_{t \succeq_p^* t'} a_1, ..., \mathcal{N}_{t \succeq_p^* t'} a_{\overline{H}}\} \oplus \{(1 - \mathcal{N}_{t \succeq_p^* t'}))a_{\overline{H}+1}, ..., (1 - \mathcal{N}_{t \succeq_p^* t'}))a_{\overline{H}}\}$ and $E = [e_1, ..., e_K]$, then

$$S(\mathcal{N}_{t \succeq_p^* t'}) = \mathrm{conv}(E),$$

or,

$$S(\mathcal{N}_{t \succeq_p^* t'}) = \mathrm{cone}(E) \cap H,$$

where

$$H = \{\pi : \mathbb{1}'_{|\pi|} \pi = T\}, \quad I := |\pi| = \sum_{t=1}^{T} I_t.$$

The following result provides an alternative representation that is useful for theoretical developments of our statistical testing procedure.

Theorem 3. *(Weyl-Minkowski Theorem for Cones) A subset $\mathcal{C}$ of $\mathbb{R}^I$ is a finitely generated cone*

$$\mathcal{C} = \{\nu_1 e_1 + ... + \nu_K e_K : \nu_h \geq 0\} \text{ for some } E = [e_1, ..., e_K] \in \mathbb{R}^{I \times K} \tag{21}$$

if and only if it is a finite intersection of closed half spaces

$$\mathcal{C} = \{t \in \mathbb{R}^I | Dt \leq 0\} \text{ for some } D \in \mathbb{R}^{m \times I}. \tag{22}$$

The expressions in (21) and (22) are called a $\mathcal{V}$-representation (as in "vertices") and a $\mathcal{H}$-representation (as in "half spaces") of $\mathcal{C}$, respectively.

See, for example, Theorem 1.3 in Ziegler (1995).[19]

In what follows we use an $\mathcal{H}$ representation of cone(E) represented by a $m \times |\pi|$ matrix D as implied by Theorem 3. The following assumptions are used for our asymptotic theory. Let $\mathbf{d}^{i,t}_{n(t)} = 1$ if the $n(t)$-th consumer in the t-th crosssection chooses patch $B^{i,t}$, and $\mathbf{d}^{i,t}_{n(t)} = 0$ if she does not. Define $\mathbf{d}^{t}_{n(t)} = [\mathbf{d}^{1,t}_{n(t)}, ..., \mathbf{d}^{I_t,t}_{n(t)}]$.

[19]See Gruber (2007), Grünbaum, Kaibel, Klee, and Ziegler (2003) and Ziegler (1995) for these results and other materials concerning convex polytopes used in this paper.

Assumption 1. For all $t = 1, ..., T$, $\frac{N_t}{N} \to \rho_t$ as $N \to \infty$, where $\rho_t > 0, 1 \le t \le T$.

Let $d_{k,i}$, $k = 1, ..., m$, $i = 1, ..., I$ denote the (k,i) element of D, then define

$$d_k(t) = [d_{k,N_1+\cdots N_{t-1}+1}, d_{k,N_1+\cdots N_{t-1}+2}, ..., d_{k,N_1+\cdots N_t}]'$$

for $1 \le t \le T$ and $1 \le k \le m$.

Assumption 2. T repeated cross-sections of random samples $\left\{ \{\mathbf{d}^{i,t}_{n(t)}\}_{i=1}^{I_t} \right\}_{n(t)=1}^{N_t}$, $t = 1, ..., T$, are observed.

The econometrician also observes the price vector p_t for each $1 \le t \le T$.

Next, we impose a mild condition that guarantees stable behavior of the statistic $\delta_N(\mathcal{N}_{t \succeq_p^* t'})$. To this end, we further specify the nature of each row of D. Recall that w.l.o.g. the first $\bar{m}$ rows of D correspond to inequality constraints, whereas the rest of the rows represent equalities. Note that the $\bar{m}$ inequalities include nonnegativity constraints $\pi_{i|t} \ge 0, 1 \le i \le I_t, 1 \le t \le T$, represented by the row of D consisting of a negative constant for the corresponding element and zeros otherwise. Likewise, the identities that $\sum_{i=1}^{I_t} \pi_{i|t}$ is constant across $1 \le t \le T$ are included in the set of equality constraints. We show in the proof that the presence of these "definitional" equalities/inequalities, which always hold by construction of $\hat{\pi}$, do not affect the asymptotic theory even when they are (close to) be binding. Define $\mathcal{K} = \{1, ..., m\}$, and let $\mathcal{K}^{\text{def}}$ be the set of indices for the rows of $D = [d_1, ..., d_m]'$ corresponding to the above nonnegativity constraints and the constant-sum constraints. Let $\mathcal{K}^R = \mathcal{K} \setminus \mathcal{K}^{\text{def}}$, so that $d_k' \pi \le 0$ represents an economic restriction if $k \in \mathcal{K}^R$.

Condition 1. For each $k \in \mathcal{K}^R$, $\text{var}(d_k(t(k))' \mathbf{d}^{t(k)}_{n(t)}) \ge \epsilon$ holds for at least one $t(k)$, $1 \le t(k) \le T$, where ϵ is a positive constant.

We now describe our bootstrap procedure, which relies on a tuning parameter τ_N chosen s.t. $\tau_N \downarrow 0$ and $\sqrt{N}\tau_N \uparrow \infty$ as in KS. Let

$$S_{\tau_N}(\mathcal{N}_{t \succeq_p^* t'}) := \{\pi = A\nu, \nu \in \mathcal{V}_{\tau_N}\}$$

where

$$\begin{aligned} \mathcal{V}_{\tau_N} := \{\nu \in \mathbb{R}^H \quad | \quad & \sum_{j=1}^{\overline{H}} \nu_j = \mathcal{N}_{t \succeq_p^* t'}, \sum_{j'=\overline{H}+1}^{H} \nu_{j'} = 1 - \mathcal{N}_{t \succeq_p^* t'}, \\ & \nu_j \ge \frac{\mathcal{N}_{t \succeq_p^* t'} \tau_N}{H}, 1 \le j \le \overline{H}, \nu_{j'} \ge \frac{(1 - \mathcal{N}_{t \succeq_p^* t'})\tau_N}{H}, \overline{H}+1 \le j' \le H\}. \end{aligned}$$

This set can be interpreted as a tightened version of the original restriction set $S(\mathcal{N}_{t \succeq_p^* t'})$. Unlike the tightening procedure proposed in KS, notice that the degree of tightening depends on the value of the hypothesized value $\mathcal{N}_{t \succeq_p^* t'}$. This "restriction-dependent tightening" is important for the validity of our procedure, which is as follows:

(i) Obtain the τ_N-*tightened restricted* estimator $\hat{\eta}_{\tau_N}$, which solves

$$\delta_{N,\tau_N}(\mathcal{N}_{t \succeq_p^* t'}) = \min_{\eta \in S_{\tau_N}(\mathcal{N}_{t \succeq_p^* t'})} N[\hat{\pi} - \eta]' \Omega [\hat{\pi} - \eta]$$

(ii) Define the τ_N-*tightened recentered* bootstrap estimators

$$\hat{\pi}^{*(r)}_{\tau_N} := \hat{\pi}^{*(r)} - \hat{\pi} + \hat{\eta}_{\tau_N}, \quad r = 1, ..., R.$$

(iii) The bootstrap test statistic is

$$\delta^{*(r)}_{N,\tau_N}(\mathcal{N}_{t\succeq^*_p t'}) = \min_{\eta \in S_{\tau_N}\mathcal{N}_{t\succeq^*_p t'}} N[\hat{\pi}^{*(r)}_{\tau_N} - \eta]'\Omega[\hat{\pi}^{*(r)}_{\tau_N} - \eta]$$

for $r = 1, ..., R$.

(iv) Use the empirical distribution of $\delta^{*(r)}_{N,\tau_N}(\mathcal{N}_{t\succeq^*_p t'}), r = 1, ..., R$ to obtain the critical value for J_N.

Note that the distribution of observations is uniquely characterized by the vector π. Let $\mathcal{P}$ denote the set of all π's that satisfy Condition 1 for some (common) value of ϵ.

Theorem 4. *Choose τ_N so that $\tau_N \downarrow 0$ and $\sqrt{N}\tau_N \uparrow \infty$. Also, let Ω be diagonal, where all the diagonal elements are positive. Then under Assumptions 1 and 2*

$$\liminf_{N\to\infty} \inf_{\pi \in \mathcal{P}\cap\mathcal{C}} \Pr\{\delta_N(\mathcal{N}_{t\succeq^*_p t'}) \le \hat{c}_{1-\alpha}\} = 1 - \alpha$$

where $\hat{c}_{1-\alpha}$ is the $1-\alpha$ quantile of δ^*_{N,τ_N}, $0 \le \alpha \le \frac{1}{2}$.

Proof. Note

$$\begin{aligned} S_{\tau_N}(\mathcal{N}_{t\succeq^*_p t'}) &= \{\pi = A\nu : \sum_{j=1}^{\overline{H}} \nu_j = \mathcal{N}_{t\succeq^*_p t'}, \sum_{j'=\overline{H}+1}^{H} \nu_{j'} = 1 - \mathcal{N}_{t\succeq^*_p t'}, \\ &\qquad \nu_j \ge \frac{\mathcal{N}_{t\succeq^*_p t'}\tau_N}{H}, 1 \le j \le \overline{H}, \nu_{j'} \ge \frac{(1-\mathcal{N}_{t\succeq^*_p t'})\tau_N}{H}, \overline{H}+1 \le j' \le H\}. \end{aligned}$$

Define

$$\tilde{\nu}_j := \begin{cases} \dfrac{\frac{\nu_j}{\mathcal{N}_{t\succeq^*_p t'}} - \frac{\tau_N}{H}}{1-\tau_n} & 1 \le j \le \overline{H} \\ \dfrac{\frac{\nu_j}{\mathcal{N}_{t\succeq^*_p t'}} - \frac{\tau_N}{H}}{1-\tau_n} & \overline{H}+1 \le j \le H, \end{cases}$$

then

$$\begin{aligned} S_{\tau_N}(\mathcal{N}_{t\succeq^*_p t'}) &= \{\pi = \mathcal{N}_{t\succeq^*_p t'}\left[(1-\tau_N)\sum_{j=1}^{\overline{H}} \tilde{\nu}_j a_j + \frac{\tau_N}{H}\overline{A}\mathbb{1}_{\overline{H}}\right] + (1-\mathcal{N}_{t\succeq^*_p t'})\left[(1-\tau_N)\sum_{j=\overline{H}}^{H} \tilde{\nu}_j a_j + \frac{\tau_N}{H}\underline{A}\mathbb{1}_{\underline{H}}\right] \\ &\qquad : \sum_{j=1}^{\overline{H}} \tilde{\nu}_j = \sum_{j'=\overline{H}+1}^{H} \tilde{\nu}_{j'} = 1, \tilde{\nu}_j \ge 0, 1 \le j \le \overline{H}, \nu_{j'} \ge 0, \overline{H}+1 \le j' \le H\} \\ &= \mathcal{N}_{t\succeq^*_p t'}\left[(1-\tau_N)\text{conv}\overline{A} \oplus \frac{\tau_N}{H}\overline{A}\mathbb{1}_{\overline{H}}\right] + (1-\mathcal{N}_{t\succeq^*_p t'})\left[(1-\tau_N)\text{conv}\underline{A} \oplus \frac{\tau_N}{H}\underline{A}\mathbb{1}_{\underline{H}}\right] \\ &= (1-\tau_N)\text{conv}\left(\{\mathcal{N}_{t\succeq^*_p t'}a_1, ..., \mathcal{N}_{t\succeq^*_p t'}a_{\overline{H}}\} \oplus \{(1-\mathcal{N}_{t\succeq^*_p t'}))a_{\overline{H}+1}, ..., (1-\mathcal{N}_{t\succeq^*_p t'}))a_{\overline{H}}\}\right) \\ &\qquad \oplus \frac{\tau_N}{H}\left[\mathcal{N}_{t\succeq^*_p t'}\overline{A}\mathbb{1}_{\overline{H}} + (1-\mathcal{N}_{t\succeq^*_p t'})\underline{A}\mathbb{1}_{\underline{H}}\right] \\ &= (1-\tau_N)\text{conv}(E) \oplus \frac{\tau_N}{H}\left[\mathcal{N}_{t\succeq^*_p t'}\overline{A}\mathbb{1}_{\overline{H}} + (1-\mathcal{N}_{t\succeq^*_p t'})\underline{A}\mathbb{1}_{\underline{H}}\right] \end{aligned}$$

For cone(E) let

$$\{\pi : D\pi \leq 0\}$$

be its $\mathcal{H}$-representation with $D \in \mathbb{R}^{m\times|\pi|}$ such that, as Lemma 4.1 in KS , $D = \begin{bmatrix} D^{\leq} \\ D^{=} \end{bmatrix}$, where the submatrices $D^{\leq} \in \mathbb{R}^{\bar{m}\times|\pi|}$ and $D^{=} \in \mathbb{R}^{(m-\bar{m})\times|\pi|}$ correspond to inequality and equality constraints, respectively.

$$\begin{aligned}
S_{\tau_N}(\mathcal{N}_{t\succeq_p^* t'}) &= \left(\mathrm{cone}(E) \oplus \frac{\tau_N}{H}\left[\mathcal{N}_{t\succeq_p^* t'}\overline{A}\mathbb{1}_{\overline{H}} + (1-\mathcal{N}_{t\succeq_p^* t'})\underline{A}\mathbb{1}_{\underline{H}}\right]\right) \cap \Delta^{H-1} \\
&= \{\pi \in \mathbb{R}^{|\pi|} : \pi - \frac{\tau_N}{H}\left[\mathcal{N}_{t\succeq_p^* t'}\overline{A}\mathbb{1}_{\overline{H}} + (1-\mathcal{N}_{t\succeq_p^* t'})\underline{A}\mathbb{1}_{\underline{H}}\right] \in \mathrm{cone}(E)\} \cap \Delta^{H-1} \\
&= \{p \in \Delta^{T-1} : D\left[p - \frac{\tau_N}{H}\left[\mathcal{N}_{t\succeq_p^* t'}\overline{A}\mathbb{1}_{\overline{H}} + (1-\mathcal{N}_{t\succeq_p^* t'})\underline{A}\mathbb{1}_{\underline{H}}\right]\right] \leq 0\} \\
&= \{p \in \Delta^{T-1} : Dp \leq -\tau_N \phi\},
\end{aligned}$$

where

$$\phi = -\frac{1}{H}D\left[\mathcal{N}_{t\succeq_p^* t'}\overline{A}\mathbb{1}_{\overline{H}} + (1-\mathcal{N}_{t\succeq_p^* t'})\underline{A}\mathbb{1}_{\underline{H}}\right].$$

for some $\phi = (\phi_1, ..., \phi_m)'$ satisifies (i) $\bar{\phi} := [\phi_1, ..., \phi_{\bar{m}}]' \in \mathbb{R}^{\bar{m}}_{++}$, and (ii) $\phi_k = 0$ for $k > \bar{m}$. Using the above notation for the $\mathcal{H}$-representation of cone(E),

$$\delta_N(\mathcal{N}_{t\succeq_p^* t'}) = \min_{\pi\in\Delta^{T-1}:D\pi\leq 0} N[\hat{\pi} - \pi]'\Omega[\hat{\pi} - \pi].$$

As in KS , define $\ell = \mathrm{rank}(D)$, and let $\ell \times m$ matrix K such that KD is a matrix whose rows consist of a basis of the row space row(D). Also let M be an $(|\pi| - \ell) \times |\pi|$ matrix whose rows form an orthonormal basis of ker$D = \ker(KD)$, and define $P = \binom{KD}{M}$. Finally, let $\hat{g} = D\hat{\pi}$.

Unlike KS , we need to augment D with a row vector of ones to properly account for the constraint $\pi \in \Delta^{T-1}$:

$$D_* := \begin{pmatrix} D \\ [1, ..., 1] \end{pmatrix}.$$

Define

$$T(x, y) = \begin{pmatrix} x \\ y \end{pmatrix}' P^{-1\prime}\Omega P^{-1}\begin{pmatrix} x \\ y \end{pmatrix}, \quad x \in \mathbb{R}^{\ell}, y \in \mathbb{R}^{|\pi|-\ell},$$

and

$$t(x) := \min_{y\in\mathbb{R}^{|\pi|-\ell}} T(x, y), \quad s(g) := \min_{\gamma=[\gamma^{\leq\prime},\gamma^{=\prime}]',\gamma^{\leq}\leq 0,\gamma^{=}=0,[\gamma',1]'\in\mathrm{col}(D_*)} t(K[g - \gamma]).$$

It is easy to see that $t : \mathbb{R}^{\ell} \to \mathbb{R}_+$ is a positive definite quadratic form. We can write

$$\begin{aligned}
\delta_N(\mathcal{N}_{t\succeq_p^* t'}) &= N \min_{\gamma=[\gamma^{\leq\prime},\gamma^{=\prime}]',\gamma^{\leq}\leq 0,\gamma^{=}=0,[\gamma',1]'\in\mathrm{col}(D_*)} t(K[\hat{g} - \gamma]) \\
&= Ns(\hat{g}) \\
&= s(\sqrt{N}\hat{g}).
\end{aligned}$$

Likewise, for the bootsgtrapped version of δ we have

$$\begin{aligned}\delta^*_{N,\tau_N}(\mathcal{N}_{t\succeq^*_p t'}) &= N \min_{\gamma=[\gamma^{\leq'},\gamma^{='}]',\gamma^{\leq}\leq 0,\gamma^{=}=0,[\gamma',1]'\in \mathrm{col}(D_*)} t(K[\hat{g}-\gamma]) \\ &= Ns(\hat{g}) \\ &= s(\varphi_{\tau_N}(\hat{\xi})+\sqrt{N}[\pi^*-\hat{\pi}]).\end{aligned}$$

where $\hat{\xi} = D\hat{\pi}/\tau_N$. The function $\varphi_N(\xi) = [\varphi^1_N(\xi), ..., \varphi^m_N(\xi)]$ for $\xi = (\xi_1, ..., \xi_m)' \in \mathrm{col}(D)$, then noting that each elements $e_j, 1 \leq j \leq K$ that defines cone(E) takes the form $\mathcal{N}_{t\succeq^*_p t'}a_h + (1 - \mathcal{N}_{t\succeq^*_p t'}))a_o, 1 \leq h \leq \overline{H}, \overline{H}+1 \leq o \leq H$, by the proof of Theorem 4.2 of KS , it follows that its k-th element φ^k_N for $k \leq \bar{m}$ satisfies

$$\varphi^k_N(\xi) = 0$$

if $|\xi^k| \leq \delta$ and $\xi^j \leq \delta, 1 \leq j \leq m$, $\delta > 0$, for large enough N and $\varphi^k_N(\xi) = 0$ for $k > \bar{m}$. The conclusion follows by Theorem 1 of Andrews and Soares (2010).

□

Theorem 4 is concerned with the validity of our bootstrap procedure concerning the restriction of the form $\mathbb{1}_{t\succeq^*_p t'}\nu = \mathcal{N}_{t\succeq^*_p t'}$. While our specific structure is useful and instructive (e.g. it allows us to provide a concrete characterization of the matrix E, which provides $S(\mathcal{N}_{t\succeq^*_p t'}) = \mathrm{conv}(E)$, as discussed above) the above procedure generalizes to a general restriction of the form $\mathcal{R}\nu = c, \mathcal{R} \in \mathbb{R}^{r\times H}, c \in \mathbb{R}^r$, once the tightening procedure is tailored appropriately to accommodate the general form. This extension is to be discussed in a subsequent version of the paper.

The crucial step in our procedure is that tightening turns non-binding inequalities in the $\mathcal{H}$-representation with small slack into binding ones but not vice versa. This feature is not universal, but it is guaranteed to work when the restriction polytope for π ($S(\mathcal{N}_{t\succeq^*_p t'})$ in the current application) satisfies a certain condition (loosely speaking, when corners of the polytope are acute), provided that the econometrician chooses Ω to be diagonal.

5. Empirical Application (Preliminary!)

We implement a test of stochastic GAPP on the 1975-1999 waves of the U.K. Family Expenditure Survey, a repeated cross-section of consumption data with reasonably high resolution. These data are a staple of the nonparametric demand literature: Blundell, Browning, and Crawford (2008), KS and Adams (2016) use the exact same data selection and in that sense are exact comparables to our implementation; Hoderlein and Stoye (2014), Kawaguchi (2017), and others also draw on the same data set. To reduce covariate variability, we restrict attention to households with cars and kids. We do not smooth data in any form: All data points are projected onto a fictitious expenditure of 1 as in Figure 5d, and choice probabilities are then estimated by corresponding sample frequencies. Computation of critical values and p-values for the test uses the "cone tightening" procedure proposed by KS and thereby is valid uniformly asymptotically over a large range of underlying parameter values.[20]

[20] In particular, since we literally use sample frequencies as estimators of probabilities, the somewhat simpler assumptions of Section 4 in KS suffice.

We implement tests for 3, 4, and 5 composite goods. The coarsest partition consisting of 3 goods precisely follows Blundell, Browning, and Crawford (2008) by considering food, services, and nondurables. Following KS, we increase the dimension of commodity space by first separating out clothing and then alcoholic beverages from the nondurables. From a purely technical point of view, this illustrates that the method is practically feasible in a 5-dimensional commodity space. With realistic sample sizes, this is not true of methods that are subject to a statistical curse of dimensionality, and nonparametric analysis in 5 dimensions is correspondingly rare in the literature.

Results are displayed in Table 1 and Table 2. Rows in these tables correspond to intervals of 6 and 7 years respectively. In principle, a single test can be conducted on the entire data from 1975 to 1999. Practically, however, we cannot conduct such a test as it is computationally infeasible. That said, there is a compelling reason to split the data and conduct the test over subsets of years as we do. Since we test the hypothesis that there is a constant distribution of preferences over time, it is sensible to consider shorter horizons as the hypothesis is unlikely to hold over the entire duration of our data. In particular, the hypothesis may fail due to large changes in the population distribution of the amount of money allocated by households towards the goods we consider (as shown in Proposition 1, small changes will not affect our test). This change may be driven by large changes in the income distribution, a fact that is true for the UK over the complete time horizon we study (Jenkins, 2016).

The closest comparison to our results are those of KS. Stochastic rationalizability is not rejected in either case. P-values are hard to compare because KS incur two additional layers of statistical noise by smoothing over expenditure (by series estimation) and by adjusting for endogeneity. Ceteris paribus, this means that the GAPP test should have higher power.

We next illustrate estimation of the proportion of consumers preferring one price vector to another one. The comparisons presented in Table 3 use data for the years 1975-1981 and 3 goods, i.e. the same data used to compute of the first row of Table 2. As estimators of bounds, we display the bounds implied by $\hat{\eta}$, i.e. the projection of empirical choice frequencies. Confidence intervals are computed by inverting the test from Section 4.2. The two year-pairs chosen, i.e. 1976-77 and 1979-80, exhibit two qualitatively different cases. For 1979-80, the fraction of the population that reveals prefers one to the other is point identified. This happens when for a particular pair of prices, the transitivity ("SARP-like") implication of GAPP does not add to its empirical content in pairwise comparison ("WARP-like") and is a somewhat frequent occurrence in the data at hand. In contrast, there is genuine partial identification for 1976-77. Our econometric methodology is robust to both scenarios, including if one does not know which obtains. To gauge the impact of tuning parameters, Confidence Intervals are presented both for $\tau = 0$ and the numerical value implied by our rule of thumb, which here is $\tau = .076$. The former intervals are not uniformly valid but give a hint as to how much distortion is caused by ensuring validity of our inference under nonstandard asymptotics. We would argue that the cost is modest in the example.

	3 goods				4 goods				5 goods			
	I	**H**	**J_N**	**p**	**I**	**H**	**J_N**	**p**	**I**	**H**	**J_N**	**p**
75-80	25	760	.337	.04	34	2571	.400	.25	34	2571	.400	.30
76-81	31	1763	.917	.34	31	1763	.698	.58	31	1763	.687	.66
77-82	26	947	.899	.55	42	7770	.651	.63	45	10772	.705	.68
78-83	29	1581	.522	.59	56	25492	.236	.91	56	25492	.329	.88
79-84	32	1862	.018	.99	106	428561	.056	.96	106	428561	.003	.999
80-85	24	625	.082	.67	106	342898	.036	.99	109	369851	.082	.96
81-86	32	1576	.088	.81	97	215891	.037	.96	101	266643	.088	.79
82-87	45	4613	.095	.91	94	229988	.043	.95	94	229988	.104	.85
83-88	31	853	.481	.61	58	30737	.043	.99	58	30737	.103	.90
84-99	18	131	.556	.48	37	3417	.232	.68	40	5019	.144	.83
85-90	10	9	.027	.69	21	138	.227	.48	21	138	.031	.85
86-91	16	53	1.42	.30	20	175	.025	.96	26	467	.019	.98
87-92	16	68	2.94	.18	29	1215	.157	.80	32	1823	.018	.97
88-93	14	37	1.51	.24	39	4928	.154	.73	42	7732	.019	.91
89-94	8	3	1.72	.21	49	16291	.004	.97	52	24538	.023	.83
90-95	8	3	0	1	66	48611	1.01	.21	87	150462	.734	.22
91-96	14	25	.313	.59	63	25946	.802	.31	69	31620	.612	.40
92-97	22	259	.700	.48	57	18616	.872	.57	60	22553	.643	.72
93-98	31	1719	.676	.60	56	25644	.904	.65	74	69780	.634	.78
94-99	40	5434	.260	.83	62	41871	.604	.74	77	90927	.488	.79

TABLE 1. Empirical results for sequences of 6 budgets. I is number of patches; H is number of rational types; J_N is value of test statistic; p is p-value.

	3 goods			
	I	**H**	**J_N**	**p**
75-81	47	30132	.965	.18
76-82	46	21363	1.58	.56
77-83	40	12826	.973	.69
78-84	44	17054	.628	.66
79-85	46	24716	.075	.96
80-86	41	11563	.096	.87
81-87	50	16259	.100	.85
82-88	50	13268	.643	.73
83-89	34	2135	.580	.66
84-90	19	131	.633	.48
85-91	21	229	1.86	.36
86-92	26	538	2.63	.28
87-93	21	250	3.27	.16
88-94	15	37	1.75	.25
89-95	11	9	1.99	.28
90-96	15	25	.367	.60
91-97	23	259	.821	.51
92-98	32	1719	.808	.61
93-99	47	22406	.755	.62

TABLE 2. Empirical results for sequences of 7 budgets. I is number of patches; H is number of rational types; J_N is value of test statistic; p is p-value.

comparison	**estimated bounds**	**CI,** $\tau = 0$	**CI,** $\tau = .076$
$p_{1977} \succ p_{1976}$	$[.119, .137]$	$[.102, .149]$	$[.094, .168]$
$p_{1976} \succ p_{1977}$	$[.843, .883]$	$[.827, .897]$	$[.821, .904]$
$p_{1980} \succ p_{1979}$	$\{.498\}$	$[.471, .524]$	$[.469, .526]$
$p_{1979} \succ p_{1980}$	$\{.464\}$	$[.436, .491]$	$[.432, .495]$

TABLE 3. Estimated bounds and Confidence Intervals for proportion of consumers preferring one price to another one. Data used are for $1975 - 1981$. Confidence intervals for $\tau = 0$ are not expected to be valid but are reported to gauge the conservative distortion due to uniform inference.

6. CONCLUSION

We developed a revealed preference analysis of a model of consumption in which the consumer maximizes utility over an observed set of purchases, taking into account a disutility of expenditure, but is not subjected to a hard budget constraint (in particular, the model generalizes quasi-linear utility). As our analysis shows, this model has bite in consumer choice settings where total expenditure is unobserved (as the data only contains information on a subset of the goods).

A useful aspect of the model is that it easily generalizes to a random utility context that arises naturally in demand analysis using repeated cross-section data but avoiding restrictions on unobserved heterogeneity. For such a setting, we show how to statistically test the model and also how to do (partially identified) welfare analysis, e.g. inference on the proportion of a population that benefit from a price change. This is illustrated with U.K. Family Expenditure Survey data. The model is not rejected, and bounds are informative. Further empirical applications are in progress.

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