The Existence of Perfect Price Indices in a Market with Heterogenous Demand

By John K.-H. Quah*

Abstract: This paper shows that the distribution of preferences among agents in a market can justify the aggregation of prices into group-price indices.

Keywords: price index, aggregation, weak separability, income effects, substitution effects, monotonicity, law of demand, weak axiom.

Author's Affiliation: Department of Economics, Oxford University.

Author's Address: St Hugh's College, Oxford. OX2 6LE. UK; john.quah@economics.ox.ac.uk

*I am grateful to the Economic and Social Research Council for providing me with financial support under its Research Fellowship Scheme (R000271171).

1. Introduction[1]

This paper deals with the problem of price aggregation in consumer theory . We assume that the commodity space has l goods, partitioned into J groups. Imagine a consumer who spends his income (assumed to be independent of prices) on these l goods; obviously, as prices vary, so will the consumer's expenditure on different goods. A set of J price indices is called *perfect* if it satisfies the following two properties. Firstly, there must be a unique price index for each group j; formally, the price index of group j is defined as a function mapping the prices of commodities in group j into a scalar. Secondly, the total expenditure on each group may be written as a function of the J group-price indices. In other words, the prices of different goods have an impact on group expenditures only insofar as it affects the values of the group-price indices; when a change in prices leaves the value of all indices unchanged, group expenditure for every group will also stay fixed.

This problem has usually been addressed in the framework of a utility-maximizing agent whose preference is weakly separable into the J groups. In that context, Gorman (1959) identified the entire class of preferences which permit the existence of perfect price indices. It turns out that the conditions needed for the existence of perfect price indices are very strong; notwithstanding this, price indices are ubiquitous in applied studies of demand, aggregate consumption and savings.

This paper is concerned with the existence of perfect price indices, but it addresses that issue in the context of a market with many agents. Of course, the results on a single agent do carry over to this case, under appropriate assumptions, but the objective of our paper is to highlight a very different scenario which will permit price aggregation for market demand.

We make no assumptions about the structure of individual preferences - in fact, agents' demand functions need not be (though they *can* be) the outcome of utility maximization. Instead we construct a model in which the *distribution of demand among agents* is such that aggregate demand permits the aggregation of prices into J perfect price indices.

The emphasis on the role of the distribution of market characteristics in obtaining various properties for market demand is not an innovation of this paper. An early paper taking such an approach is Hildenbrand (1983). That paper identified conditions on the income distribution which are sufficient to guarantee that market demand obeys the law of demand. In that model, individual demand must obey the weak axiom, but it does not need to obey the law of demand. So an aggregate property is created which does not hold at the individual level.

Another model which similarly emphasizes the distribution of market characteristics is Grandmont (1992). Grandmont constructed a model in which the distribution of preferences could be chosen such that market demand becomes increasingly Cobb-Douglas-like, in the sense that the expenditure on each good becomes increasingly independent of all prices. Once again, Cobb-Douglas behavior need not hold at the individual level, but is 'created' through aggregation across the heterogenous agents in the market.

The model in this paper is based on Grandmont's in the sense that, following Grandmont, we model demand heterogeneity through the use of household equivalence scales (or, following the terminology of Grandmont's paper, affine transformations). Indeed our model could be thought of as a generalization of his model since having aggregate Cobb-Douglas behavior, in which the expenditure on every good is a constant independent of price, is clearly an extreme situation in which price aggregation is possible. In this paper we construct a model of demand heterogeneity in which perfect price indices exist for market demand, but where the heterogeneity is (in some precise sense) not so great as to lead to aggregate Cobb-Douglas behavior.

2. A Price Aggregation Theorem

We consider a market in which consumers choose from l commodities, labelled 1,2,...,l. These commodities are divided into J *commodity groups*; the typical group j, which we denote by G^j, consists of $k^j \geq 1$ commodities. So the first group consists of the first k^1 commodities, i.e., $G^1 = \{1, 2, ..., k^1\}$; $G^2 = \{k^1 + 1,\ k^1 + 2, ..., k^1 + k^2\}$, etc. Clearly, $\sum_{j=1}^{J} k^j = l$.

Let a^j be a vector in R^l such that $a_i^j > 0$ if and only if i is in G^j and $\sum_{i=1}^{l} a_i^j = 1$. We call any vector with these properties a *weighting vector for the commodity group G^j*. We refer to the function $P^{a^j} : R_{++}^l \rightarrow R_{++}$, defined by

$$P^{a^j}(\hat{q}) \quad = \quad \prod_{i \in G^j} \hat{q}_i^{a_i^j} \tag{1}$$

as the *price index for G^j with weighting vector a^j*. The motivation for this terminology should be clear once we interpret $\hat{q}$ as the vector of prices for the l commodities, in which case P^{a^j} amalgamates the prices in commodity group G^j into a single number, with different commodities entering into the index in a way determined by the vector a^j. We call a set A in R_{+}^l a *complete set of weighting vectors* if it consists of J weighting vectors, with exactly one weighting vector for each commodity group. So we may write $A = \{a^j\}_{1 \leq j \leq J}$.

We now turn to the description of the agents in the market, beginning with the description of the generator. (The last term may sound a little mysterious at this point, but its motivation will be clear as we describe the model.) Let $\hat{f} : R^l_{++} \to R^l_+$ be a function satisfying $\hat{q} \cdot \hat{f}(\hat{q}) = 1$. We interpret $\hat{f}(\hat{q})$ as the demand (over l commodities) of an agent with an income normalized at 1 and facing the price vector $\hat{q}$. Given $\hat{f}$, we can define $\hat{w} : R^l_{++} \to R^l_+$ by $\hat{w}_i(\hat{q}) = \hat{q}_i \hat{f}_i(\hat{q})$. In other words, $\hat{w}_i(\hat{q})$ is the expenditure devoted to good i at price $\hat{q}$. It is more convenient for our purposes to work with log prices, so we define $\bar{w} : R^l \to R^l_+$ by $\bar{w}(q) = \hat{w}(e^{q_1}, e^{q_2}, ..., e^{q_l})$. Note that $\sum_{i=1}^l \bar{w}(q) = 1$; it also follows, since $\hat{f}$ and hence $\bar{w}$ is uniformly bounded below by 0, that $\bar{w}$ is uniformly bounded above. We shall refer to $\bar{w}$ as the *expenditure share function* of $\hat{f}$.

Given $\hat{f}$ and given θ in R^l, we may define $f_\theta : R^l_{++} \to R^l_+$ by

$$f_{\theta\, i}(\hat{q}) \quad = \quad e^{\theta_i} \hat{f}_i(\hat{q}_1 e^{\theta_1}, \hat{q}_2 e^{\theta_2}, ..., \hat{q}_l e^{\theta_l}) \tag{2}$$

for $i = 1, 2, ..., l$. Following the literature (see Grandmont (1992)), we will refer to f_θ as an *affine transformation of* $\hat{f}$. It is trivial to check that the expenditure share function of f_θ, which we denote by w_θ, is related to $\bar{w}$ by $w_\theta(q) = \bar{w}(\theta + q)$: so w_θ is just a translation of $\bar{w}$ by the vector θ.

The special value of affine transformations lies in the fact that it preserves standard rationality properties that $\hat{f}$ might satisfy. In particular the following claims are easy to check: (a) if $\hat{f}$ is generated by a utility function, so is f_θ; (b) if $\hat{f}$ satisfies the weak axiom, so will f_θ; and (c) if $\hat{f}$ satisfies the law of demand, so will f_θ.[2]

Let Θ be a subspace of R^l. Consider the set $\mathcal{W}_\Theta = \{w_\theta : \theta \in \Theta\}$. Clearly it is reasonable to speak of this set as being 'generated' by $\bar{w}$, which explains our earlier reference to $\hat{f}$ (from which $\bar{w}$ is derived) as the 'generator'. $\mathcal{W}_\Theta$ is a subset of $C(R^l, R^l)$, the set of all continuous functions from R^l to R^l. We endow $C(R^l, R^l)$ with the topology of uniform convergence on compact sets. (Indeed, with this topology, $C(R^l, R^l)$ is also metrizable.) We denote the closure of $\mathcal{W}_\Theta$ by $\overline{\mathcal{W}}_\Theta$.

Clearly for each w in $\overline{\mathcal{W}}_\Theta$, we have $\sum_{i=1}^{l} w_i(\hat{q}) = 1$ and $w(q) > 0$ at all q, so we may consider w as an expenditure share function, with the corresponding demand function denoted by $f_w : R^l_{++} \to R^l_{+}$, where $f_{w\,i}(\hat{q}) = w_i(\ln \hat{q}_1, \ln \hat{q}_2, ..., \ln \hat{q}_l)/\hat{q}_i$. Clearly, if $w = w_\theta$ for some θ, then $f_w = f_\theta$. (The notation here is less than ideal, but the meaning should be clear.) When $\hat{f}$ satisfies a rationality property like those in (a), (b) or (c), f_w need not satisfy the property exactly, but on any compact set of strictly positive prices, it will be arbitrarily close to a function $f_{w'}$ which *does* satisfy the property. This follows easily from the fact that the members of $\overline{\mathcal{W}}_\Theta$ are either translations of $\bar{w}$ or arbitrarily close, on compact sets, to such translations of $\bar{w}$. So by ensuring that the generating demand function $\hat{f}$ has the rationality property we want, we can guarantee that all the demand functions corresponding to expenditure share functions in $\overline{\mathcal{W}}_\Theta$ will share that same property, if not exactly then at least approximately. In this sense, any market where agents have expenditure share functions drawn from $\overline{\mathcal{W}}_\Theta$ can be made compatible with rationality.

Consider a market (or more accurately, that slice of the market) consisting of all agents with the same income (which we normalize at 1) and possibly heterogenous demand behavior. Given a complete set of weighting vectors, $A = \{a^j\}_{1 \le j \le J}$, we define the subspace $\Theta(A) = \{\theta \in R^l : \theta \perp a^j \text{ for all } j = 1, 2, ...J\}$. We assume that all the agents in the market have expenditure share functions drawn from $\overline{\mathcal{W}}_{\Theta(A)}$. Let ν be a probability measure on $\overline{\mathcal{W}}_{\Theta(A)}$; for this measure, the *mean expenditure share function* of the market, $W_\nu : R^l_{++} \to R^l_+$, is given by

$$(3) \qquad W_\nu(\hat{q}) = \int_{\overline{\mathcal{W}}_{\Theta(A)}} \Psi(\ln \hat{q}, w) d\nu.$$

(Note, firstly, that the terminology here is slightly inconsistent: we refer to W_ν as the mean expenditure share function even though it is defined on prices $\hat{q}$ rather than log-prices. Secondly, $\ln \hat{q}$ refers to the vector of log-prices, i.e., $(\ln \hat{q}_1, \ln \hat{q}_2, ..., \ln \hat{q}_l)$. Finally, the function $\Psi : R^l \times \overline{\mathcal{W}}_{\Theta(A)} \to R^l$ is defined by $\Psi(q, w) = w(q)$.)

We say that the collection of indices $\{P^{a^j}\}_{j \in J}$ is *a set of perfect price indices* (or a perfect index set) for W_ν if there exists a function $W^A_\nu : R^J_{++} \to R^J_+$ such that, for all groups G^j,

$$(4) \qquad \sum_{i \in G^j} W_{\nu\, i}(\hat{q}) = W^A_{\nu\, j}(P^{a^1}(\hat{q}), P^{a^2}(\hat{q}), ..., P^{a^J}(\hat{q})).$$

We shall refer to W^A_ν as *the group expenditure function associated with* W_ν *and the perfect index set* $\{P^{a^j}\}_{j \in J}$. It is clear that a necessary and sufficient condition for $\{P^{a^j}\}_{j \in J}$ to be a perfect index set is that a change in prices which preserves the value of all indices must leave all group expenditures unchanged. Formally, we require W_ν to obey the following condition:

(C) *if prices $\hat{q}'$ and $\hat{q}''$ satisfy $P^{a^j}(\hat{q}') = P^{a^j}(\hat{q}'')$ for all j, then*

$$(5) \qquad \sum_{i \in G^j} W_{\nu\, i}(\hat{q}') = \sum_{i \in G^j} W_{\nu\, i}(\hat{q}'') \text{ for all commodity groups } G^j.$$

If this is true, we may construct W^A_ν along the following lines. Let ϵ be the vector $(1, 1, ..., 1)$ in R^l_{++}; for any p in R^J_{++}, let λ^j be the unique scalar such that $P^{a^j}(\lambda^j \epsilon) = p_j$. We define the function $L : R^J_{++} \to R^l_{++}$ by $L_i(p) = \lambda^j$ if i is in G^j and the function W^A_ν by

$$W^A_{\nu\,j}(p) \quad = \quad \sum_{i \in G^j} W_{\nu\,i}(L(p)). \tag{6}$$

We are now ready to state the main result of this paper. It says that for a given a set of indices $\{P^{a^j}\}_{j \in J}$, there exists some distribution ν over $\overline{\mathcal{W}}_{\Theta(A)}$ such that $\{P^{a^j}\}_{j \in J}$ is a perfect index set for the mean expenditure share function W_ν. The theorem establishes this by establishing a condition (C′) which is stronger than condition (C).

THEOREM 1: *Let $A = \{a^j\}_{1 \le j \le J}$ be a complete set of weighting vectors and suppose that the expenditure share function $\bar{w} : R^l \to R^l_+$ is uniformly continuous. Then $\overline{\mathcal{W}}_{\Theta(A)}$ is a compact subset of $C(R^l, R^l)$ and there exists a Borel probability measure ν^* on $\overline{\mathcal{W}}_{\Theta(A)}$ such that W_{ν^*} satisfies the following condition:*

(C′) *if prices $\hat{q}'$ and $\hat{q}''$ satisfy $P^{a^j}(\hat{q}') = P^{a^j}(\hat{q}'')$ for all j, then*

$$W_{\nu^*}(\hat{q}') \quad = \quad W_{\nu^*}(\hat{q}''). \tag{7}$$

Consequently, $\{P^{a^j}\}_{j \in J}$ is a perfect index set for W_{ν^*}.

It is worthwhile at this point to compare Theorem 1 with Grandmont's (1992) theorem (rather more precisely, Giraud and Quah's (2003, Example 1) exact version of his approximate result). We define $\mathcal{W} = \{w_\theta : \theta \in R^l\}$ and denote its closure in $C(R^l, R^l)$ by $\overline{\mathcal{W}}$.

THEOREM 2: *Suppose that the expenditure share function $\bar{w} : R^l \to R^l_+$ is uniformly continuous. Then $\overline{\mathcal{W}}$ is a compact subset of $C(R^l, R^l)$ and there exists a Borel probability measure ν^{**} on $\overline{\mathcal{W}}$ such that $W_{\nu^{**}}$ satisfies the following condition:*

(C″) *for any two prices $\hat{q}'$ and $\hat{q}''$,we have $W_{\nu^{**}}(\hat{q}') = W_{\nu^{**}}(\hat{q}'')$.*

In other words, at all prices, $W_{\nu^{**}}$ *equals some constant vector* $(b_1, b_2, ..., b_l)$*, so that* $W_{\nu^{**}}$ *coincides with the expenditure share function induced by the Cobb-Douglas utility function* $U : R^l_{++} \to R$*, where* $U(z) = \prod_{i=1}^{l} z_i^{b_i}$.

The difference between Theorems 1 and 2 turns on the difference between (C′) and (C″). The latter asks for a market in which W_ν is constant at *all* prices, whereas in the former we only require that W_ν be constant on *certain subsets* of prices; formally, (C′) requires W_ν to be constant in each *index-invariant set*, a typical set being

$$\{\hat{q} \in R^l_{++} : P^{a^j}(\hat{q}) = k^j \text{ for } j = 1, 2, ...l\},$$

where $k^j > 0$ $(j = 1, 2, ...J)$. Under standard conditions, sets of this form are $(l - J)$-dimensional manifolds in R^l. To obtain the stronger property (C″), Theorem 2 requires correspondingly stronger assumptions on the heterogeneity of demand behavior: the probability measure in Theorem 2 is defined on $\overline{\mathcal{W}}$, whereas in Theorem 1 it is defined on $\overline{\mathcal{W}}_{\Theta(A)}$, and of course, $\overline{\mathcal{W}}_{\Theta(A)} \subset \overline{\mathcal{W}}$.

3. Proving Theorem 1

Our proof of Theorem 1 will rely on a mathematical result first established in Giraud and Quah (2003).

Let $\mathcal{P}$ be a σ-compact metric space; by this we mean that there is a sequence of compact sets $\{\mathcal{P}_n\}_{n\geq 1}$ such that $\cup_{n=1}^{\infty}\mathcal{P}_n = \mathcal{P}$ and $\mathcal{P}_n \subset \mathcal{P}^o_{n+1}$, where the latter refers to the interior of $\mathcal{P}_{n+1}$.[3] Let $\bar{s} : \mathcal{P} \times R^l \to R^m$ be a continuous function. Given t in R^l, we can define s_t, a function from $\mathcal{P} \times R^l$ to R^m by $s_t(p, x) = \bar{s}(p, x + t)$. We denote by $\mathcal{S}$ the set $\{s_t : t \in R^l\}$. This is a subset of $C(\mathcal{P} \times R^l, R^m)$, the set of continuous functions from $\mathcal{P} \times R^l$ to R^m.

We endow $C(\mathcal{P} \times R^l, R^m)$ with the topology of uniform convergence on compact sets, so it makes sense to speak of the closure of $\mathcal{S}$, which we shall denote by $\overline{\mathcal{S}}$.

THEOREM 3: *Suppose that* $\mathcal{P}$ *is a* σ*-compact metric space and that* $\bar{s} : \mathcal{P} \times R^l \to R^m$ *is uniformly continuous, with* $\bar{s}(p, \cdot)$ *uniformly bounded for every* p*. Then* $\overline{\mathcal{S}}$ *is a compact subset of* $C(\mathcal{P} \times R^l, R^m)$ *and there exists a Borel probability measure* μ^* *on* $\overline{\mathcal{S}}$ *with the following property: given any continuous function* $H : V \times R^m \to R^n$*, where* V *is a metric space, there exists a continuous function* $H^* : V \times P \to R^n$ *such that for all* x *in* R^l*,*

$$(8) \qquad \int_{\overline{\mathcal{S}}} H(v, \Phi(p, x, s)) d\mu^* \;=\; \int_{\overline{\mathcal{S}}} H(v, s(p, x)) d\mu^* = H^*(v, p).$$

(Note that the function $\Phi : \mathcal{P} \times R^l \times \overline{\mathcal{S}} \to R^m$ is defined by $\Phi(p, x, s) = s(p, x)$.)

The proof of Theorem 3 can be found in Giraud and Quah (2003).[4] The next result is an obvious corollary.

COROLLARY 1: *Suppose that* $\mathcal{P}$ *is a* σ*-compact metric space and that* $\bar{s} : \mathcal{P} \times R^l \to R^m$ *is uniformly continuous, with* $\bar{s}(p, \cdot)$ *uniformly bounded for every* p*. Then* $\overline{\mathcal{S}}$ *is a compact subset of* $C(\mathcal{P} \times R^l, R^m)$ *and there exists a Borel probability measure* μ^* *on* $\overline{\mathcal{S}}$ *such that for all* x *in* R^l*,*

$$(9) \qquad \int_{\overline{\mathcal{S}}} \Phi(p, x, s)) d\mu^* \;=\; \int_{\overline{\mathcal{S}}} s(p, x) d\mu^* = H^*(p).$$

The interpretation of Corollary 1 is as follows. $\Phi(p, x, s) = s(p, x)$, so the average value of Φ at (p, x) for some distribution on s will in general depend on both p and x. But the corollary says that there exists at least one distribution on s such that the average value of Φ ceases to depend on x and is only dependent on p. This property will be crucial to our proof of Theorem 1.

Notice also that Theorem 2 is just a special case of Corollary 1. If $\bar{s}$ does not depend on p and is just a uniformly continuous and bounded function (of x) from R^l to R^m, then Corollary 1 says that there is some probability measure μ^{**} on $\overline{\mathcal{S}}$ such that $\int_{\overline{\mathcal{S}}} \Phi(x,s))d\mu^{**}$ is a constant. To obtain Theorem 2, set $m = l$, identify $\mathcal{W}$ with $\mathcal{S}$, s with w, Φ with Ψ, and x with the log-price $\ln \hat{q}$.

This leaves us with the proof of Theorem 1.

PROOF OF THEOREM 1: Let $\hat{a}^j > 0$ be the unit vector parallel to a^j. To the vectors $\hat{a}^1$, $\hat{a}^2$,...,$\hat{a}^J$, add unit vectors u^{J+1}, u^{J+2},... u^l such that these vectors together form an orthonormal basis for R^l. Note that the subspace $\Theta(A)$ is generated by $\{u^{J+1}, u^{J+2}, ..., u^l\}$. To each log-price q in R^l, we can associate $(p(q), x(q))$ in $R^J \times R^{l-J}$ where $(p(q), x(q))$ are the coefficients arising from the decomposition of y on the orthonormal basis. So we have $p_i(q) = q \cdot \hat{a}^i$ for all $i = 1, 2, ..., J$ and $x_i(q) = q \cdot u^{J+i}$, for $i = 1, 2, ..., l-J$. Clearly, the map from q to $(p(q), x(q))$ is linear (and therefore continuous), one-to-one, and onto; we shall denote this map by χ.

We define $\bar{s} : R^J \times R^{l-J} \to R^l_+$ by $\bar{s}(p,x) = \bar{w}(\chi^{-1}(p,x))$. The boundedness and uniform continuity of $\bar{w}$ means that $\bar{s}$ will inherit the same properties. Therefore, by Theorem 3 (and with $\mathcal{P} = R^J$, which is indeed σ-compact), $\overline{\mathcal{S}}$ is a compact set. Define the map $M : \overline{\mathcal{S}} \to C(R^l, R^l)$ by $M(s)\,(q) = s(\chi(q))$. It is straightforward to check that this map is one-to-one; it is also continuous, so that $M(\overline{\mathcal{S}})$ is a compact set in $C(R^l, R^l)$.

We claim that $M(\overline{\mathcal{S}}) = \overline{\mathcal{W}}_{\Theta(A)}$, so that the latter is also compact (thus establishing the first claim of Theorem 1). To see this, note firstly that $M(\mathcal{S}) = \mathcal{W}_{\Theta(A)}$ which follows from that fact that $M(s_t) = w_\theta$ where $\theta = \chi^{-1}(0,t)$. Therefore, $\overline{M(\mathcal{S})} = \overline{\mathcal{W}}_{\Theta(A)}$; we need only show that $\overline{M(\mathcal{S})} = M(\overline{\mathcal{S}})$. That $M(\overline{\mathcal{S}}) \subset \overline{M(\mathcal{S})}$ follows immediately from the definitions; furthermore, because $\bar{S}$ is compact, we also have $\overline{M(\mathcal{S})} \subset M(\overline{\mathcal{S}})$.

Let μ be some probability measure on $\overline{\mathcal{S}}$. This induces a probability measure ν on $\overline{\mathcal{W}}_{\Theta(A)}$ via M. Then

$$\begin{aligned} \int_{\overline{\mathcal{S}}} \Phi(p,x,s))d\mu &= \int_{\overline{\mathcal{W}}_{\Theta_A}} \Phi(p,x,M^{-1}(w))d\nu \\ &= \int_{\overline{\mathcal{W}}_{\Theta_A}} \Psi(\chi^{-1}(p,x),w)d\nu. \end{aligned} \tag{10}$$

By Theorem 2, there is μ^* (and correspondingly ν^*) such that

$$\int_{\overline{\mathcal{S}}} \Phi(p,x,s))d\mu^* = \int_{\overline{\mathcal{W}}_{\Theta_A}} \Psi(\chi^{-1}(p,x),w)d\nu^* \tag{11}$$

is independent of x.

Let $\hat{q}'$ and $\hat{q}''$ be two prices (with corresponding log-prices q' and q'') which induce exactly the same index values for all commodity groups, i.e., $P^{a^j}(\hat{q}') = P^{a^j}(\hat{q}'')$ for all j. One can check that this guarantees that $p(q') = p(q'')$ though $x(q')$ need not equal $x(q'')$. By (11) and the property of μ^*,

$$\int_{\overline{\mathcal{W}}_{\Theta_A}} \Psi(q',w)d\nu^* = \int_{\overline{\mathcal{S}}} \Phi(p(q'),x(q'),s))d\mu^* = \int_{\overline{\mathcal{S}}} \Phi(p(q''),x(q''),s))d\mu^* = \int_{\overline{\mathcal{W}}_{\Theta_A}} \Psi(q'',w)d\nu^*.$$

So we have established the W_{ν^*} obeys condition (C$'$). QED

4. Price Aggregation in an Income-Heterogenous Market

Theorem 1 considered price aggregation in the context of a market where agents have heterogenous demand functions but the same income, normalized at 1. In this section, we shall consider this problem in a market where agents are permitted to have different income levels. An issue which immediately arises in this context is the joint distribution of demand functions (or preferences) and income. A natural benchmark, which we will adopt, is to assume that they are independent. We will now describe such a model of the market.

Firstly, we consider market demand conditional on income $y = 1$. As in Section 2, we assume that expenditure share functions are drawn from $\overline{\mathcal{W}}_{\Theta(A)}$ according to the probability measure ν. Hence the mean expenditure on good i at price $\hat{q}$ is $\int_{\bar{\mathcal{W}}_{\Theta(A)}} \Phi_i(\ln \hat{q}, w) d\nu$. At this point, we make two crucial and standard assumptions. We assume that all demand functions are zero-homogenous in price and income and that the distribution of demand functions (or more accurately, of their corresponding expenditure share functions) is independent of income. It follows that the mean expenditure on good i conditional on income $y = y'$ is $\int_{\bar{\mathcal{W}}_{\Theta(A)}} \Phi_i(\ln(\hat{q}/y'), w)\, y'\, d\nu$. We denote by $W_{(\nu,\sigma)} : R^l_{++} \to R^l_{+}$ the mean expenditure in the market when demand behavior is distributed according to the probability measure ν on $\overline{\mathcal{W}}_{\Theta(A)}$ and the income distribution is governed by the Borel probability measure σ on R_+. Then

$$W_{(\nu,\sigma)\,i}(\hat{q}) = \int_{R_+} \left[\int_{\overline{\mathcal{W}}_{\Theta(A)}} \Phi_i(\ln(\hat{q}/y), w)\ d\nu \right] y\, d\sigma. \tag{12}$$

The next result shows that there is indeed some probability measure ν^* on the demand functions such that $\{P^{a^j}\}_{j\in J}$ is a perfect index set for $W_{(\nu^*,\sigma)}$. In essence, the result stems from the fact that the price index functions P^{a^j} are all homogenous of degree 1. It is also worth noting that ν^* does not depend on σ; in other words, $\{P^{a^j}\}_{j\in J}$ will remain a perfect index set even if there is a change in the distribution of income, provided the distribution of expenditure share functions (and thus, assuming rationality, the corresponding preferences) is unchanged at ν^*.

COROLLARY 2: *Let* $A = \{a^j\}_{1 \le j \le J}$ *be a complete set of weighting vectors and suppose that the expenditure share function* $\bar{w} : R^l \to R^l_+$ *is uniformly continuous. Then* $\overline{\mathcal{W}}_{\Theta(A)}$ *is a compact subset of* $C(R^l, R^l)$ *and there exists a Borel probability measure* ν^* *on* $\overline{\mathcal{W}}_{\Theta(A)}$ *such that* $\{P^{a^j}\}_{j \in J}$ *is a perfect index set for* $W_{(\nu^*,\sigma)}$ *(as defined by (12)), for any Borel probability measure* σ *on* R_+*.*

Proof: We argue that if demand behavior is distributed according to ν^* (as defined in Theorem 1) then $\{P^{a^j}\}_{j \in J}$ is a perfect index set for $W_{(\nu^*,\sigma)}$, with the corresponding group expenditure function $W^A_{(\nu^*,\sigma)} : R^J_{++} \to R^J_+$ defined by $W^A_{(\nu^*,\sigma)\,j}(p) = \int_{R_+} W^A_{\nu^*\,j}(p/y) y \, d\sigma$, where $W^A_{\nu^*}$ is given by (6). This is true since

$$\begin{aligned}
\sum_{i \in G^j} W_{(\nu^*,\sigma)\,i}(\hat{q}) &= \int_{R_+} \left[\int_{\bar{\mathcal{W}}_{\Theta(A)}} \sum_{i \in G^j} \Phi_i(ln\,(\hat{q}/y), w) \; d\nu^* \right] y \, d\sigma \\
&= \int_{R_+} W^A_{\nu^*\,j} \left(P^{a^1}(\hat{q}/y), P^{a^2}(\hat{q}/y), ..., P^{a^J}(\hat{q}/y) \right) y \, d\sigma \\
&= \int_{R_+} W^A_{\nu^*\,j} \left(\frac{P^{a^1}(\hat{q})}{y}, \frac{P^{a^2}(\hat{q})}{y}, ..., \frac{P^{a^J}(\hat{q})}{y} \right) y \, d\sigma.
\end{aligned}$$

The first equality follows from (12) while the second arises from the fact that $W^A_{\nu^*}$ is the group expenditure function associated with W_{ν^*} and $\{P^{a^j}\}_{j \in J}$. The crucial third equality follows from a nice property of the price indices we have constructed and which we have not used until now: the functions P^{a^j} are all homogeneous of degree 1. QED

References

Giraud, G. and J. K.-H. Quah (2003): Exact Homothetic or Cobb-Douglas Demand Behavior Through Aggregation," *Contributions to Theoretical Economics*, Vol. 3, no. 1, Article 8.

Gorman, W. M. (1959): "Separable Utility and Aggregation," *Econometrica*, Vol. 27, 469-481.

Grandmont, J. M. (1992): "Transformations of the commodity space, behavioral heterogeneity, and the aggregation problem," *Journal of Economic Theory*, Vol. 57, 1-35.

Hildenbrand, W. (1983): "On the Law of Demand," *Econometrica*, Vol. 51, 997-1019.

Mas-Colell, A., M. D. Whinston, and J. R. Green (1995): *Microeconomic Theory.* Oxford: Oxford University Press.

Footnotes

1. This whole paper is very much work in progress. This is especially true of the Introduction which, amongst other defects, is very under-referenced. Suggestions, especially relating to connections with the existing literature, are more than welcome.

2. A function $f : R^l_{++} \times R_+ \to R^l_{++}$ satisfies the weak axiom if $p \cdot f(p', w') \leq p \cdot f(p, w)$ implies $p' \cdot f(p, w) > p' \cdot f(p', w')$. It satisfies monotonicity or the law of demand if $(p - p') \cdot (f(p, w) - f(p', w)) < 0$. For a discussion of these concepts see Mas-Colell et al (1996). In the case of (a), if $\bar{f}$ is generated by u, then f_t is generated by u_t, where $u_t(x) = u(e^{-t^1}x^1, e^{-t^2}x^2, ..., e^{-t^l}x^l)$.

3. Note that σ-compact sets includes compact sets and singletons. If a set P is compact, simply take $P_n = P$ for all n. Then $P_n^0 = P$ and $P_n \subset P_{n+1}^0$ is trivially true.

4. The theorem stated here is stronger than Theorem 3 in Giraud and Quah (2003). In that theorem, it is suggested that the choice of μ^* depends on H when in fact it does not: a cursory examination of their proof will reveal that there is some μ^* for which (7) is true for *any* continuous function H.