Notation

The set

\[ R^l = \{ x = (x_1, x_2, \ldots, x_l) : x_i \in R \} \]

is known as the Euclidean space. We denote

\[ R^l_+ = \{ x \in R^l : x_i \geq 0 \ \forall \ i \} \]
\[ R^l_{++} = \{ x \in R^l : x_i > 0 \ \forall \ i \} \]

We sometimes refer to \( R^l_+ \) as the positive orthant.

For vectors \( x \) and \( y \) in \( R^l \), we say

\( x \geq y \) if \( x_i \geq y_i \) for all \( i \) and
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\[ x \geq y \text{ if } x_i \geq y_i \text{ for all } i \text{ and } x > y \text{ if } x_i \geq y_i \text{ for all } i \text{ and } x \neq y. \]
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For vectors $x$ and $y$ in $R^l$, we say

$x \geq y$ if $x_i \geq y_i$ for all $i$ and $x > y$ if $x_i \geq y_i$ for all $i$ and $x \neq y$.

Finally, $x \gg y$ if $x_i > y_i$ for all $i$. 
Notation

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Finally, \( x \gg y \) if \( x_i > y_i \) for all \( i \).

Examples: \((1, 3, 3) > (1, 2, 3)\) and \((1, 3, 3) \gg (0, 2, 1)\).
Demand Theory

Assume that there are $l$ commodities and that the consumption space is $R^l_+$ (the positive orthant).

Agent has the utility function $U : R^l_+ \to \mathbb{R}$. 
Demand Theory

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Agent has the utility function $U : R^l_+ \rightarrow R$.

Recall the following:

$U$ is said to be increasing if $x' \gg x$ implies $U(x') > U(x)$.

$U$ is strictly increasing if $x' > x$ implies $U(x') > U(x)$.
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\( U \) is **quasiconcave** if, for any \( \alpha \), the set \( \{ x \in R^l_+ : U(x) \geq \alpha \} \) is a convex set. Equivalently, whenever \( U(x') \geq \alpha \) and \( U(x) \geq \alpha \), then \( U(tx' + (1 - t)x) \geq \alpha \) for any \( t \in [0, 1] \).
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$U$ is **strictly quasiconcave** if, whenever $U(x') \geq \alpha$ and $U(x) \geq \alpha$, then $U(tx' + (1-t)x) > \alpha$ for any $t \in (0, 1)$. 
Demand Theory

At price (vector) \( p = (p_1, p_2, ..., p_l) \) in \( R^{l}_{++} \) (we also write \( p \gg 0 \)) and income \( w > 0 \), the agent’s budget set is

\[
B(p, w) = \{ x \in R^l_+ : p \cdot x \leq w \}.
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At price (vector) \( p = (p_1, p_2, \ldots, p_l) \) in \( R_{++}^l \) (we also write \( p \gg 0 \)) and income \( w > 0 \), the agent’s budget set is

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A commodity bundle \( x^* \) is a demand bundle of the agent at \( (p, w) \) if \( x^* \) maximizes utility in \( B(p, w) \); formally,

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x^* \in \argmax_{x \in B(p, w)} U(x).
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At price (vector) \( p = (p_1, p_2, \ldots, p_l) \) in \( \mathbb{R}^{l+} \) (we also write \( p \gg 0 \)) and income \( w > 0 \), the agent’s budget set is

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Note that \( B(tp, tw) = B(p, w) \) for any \( t > 0 \), so

\[ \arg\max_{x \in B(p, w)} U(x) = \arg\max_{x \in B(tp, tw)} U(x). \]
Demand Theory

Let \( x^* \in \text{argmax}_{x \in B(p,w)} U(x) \).

Quite easy to see that

(A) if \( U \) is increasing, then \( p \cdot x^* = w \). In this case, we say that the agent’s demand obeys the budget identity.

(B) if \( U \) is strictly quasiconcave, then \( \text{argmax}_{x \in B(p,w)} U(x) \) has at most one element.

(C) \( \text{argmax}_{x \in B(p,w)} U(x) \) is nonempty if \( U \) is a continuous function.

(This is a straightforward consequence of Weierstrass Theorem\(^a\))

\(^a\) **Weierstrass Theorem**: Suppose that \( K \) is a compact set in \( \mathbb{R}^l \) and \( F : K \to \mathbb{R} \) a continuous function. Then \( \text{arg max}_{x \in K} F(x) \) is nonempty.
Demand Theory

Result below summarizes (A)-(C) and more.

Proposition: Suppose that the utility function $U : \mathbb{R}^l_+ \rightarrow \mathbb{R}$ is (P1) continuous, (P2) strictly increasing, and (P3) strictly quasiconcave.

Then for any $(p, w)$ in $\mathbb{R}^l_+ \times \mathbb{R}^*_+$, there exists a unique element $x^*$ in $\arg\max_{x \in B(p, w)} U(x)$. 

Lectures on General Equilibrium Theory
Demand Theory

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**Proposition:** Suppose that the utility function \( U : \mathbb{R}_{++}^{l} \rightarrow \mathbb{R} \) is
(P1) continuous, (P2) strictly increasing, and (P3) strictly quasiconcave.

Then for any \((p, w)\) in \( \mathbb{R}_{++}^{l} \times \mathbb{R}_{++} \), there exists a unique element \( x^* \) in
\[ \arg\max_{x \in B(p, w)} U(x). \]

The function \( \bar{x} : \mathbb{R}_{++}^{l} \times \mathbb{R}_{++} \rightarrow \mathbb{R}_{+} \) mapping
\((p, w)\) to \( \bar{x}(p, w) = \arg\max_{x \in B(p, w)} U(x) \) has the following properties:

(a) it is continuous,
(b) it obeys the budget identity [i.e., \( p \cdot \bar{x}(p, w) = w \)],
(c) it is zero-homogeneous, [i.e. \( \bar{x}(tp, tw) = \bar{x}(p, w) \) for any \( t > 0 \)] and
(d) it obeys this boundary condition: if \( (p^n, w^n) \rightarrow (\bar{p}, \bar{w}) \) such that \( \bar{w} > 0 \) and \( I = \{i : \bar{p}_i = 0\} \) is nonempty, then

\[ \sum_{i \in I} \bar{x}^a_i (p^n, w^n) \rightarrow \infty. \]
Exchange Economy

Assume that there is a finite set $A$ of agents.

Agent $a \in A$ has the utility function $U^a : R_+^l \rightarrow R$ and endowment $\omega^a = (\omega_1^a, \omega_2^a, ..., \omega_l^a)$ in $R_+^l$.

We require $U^a$ to obey (P1), (P2), and (P3), and that aggregate endowment

$$\bar{\omega} = \sum_{a \in A} \omega^a \gg 0.$$
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Lectures on General Equilibrium Theory
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In an exchange economy, income is determined by the prevailing price $p$. With an endowment of $\omega^a$, agent $a$’s income $w^a = p \cdot \omega^a$. 

Lectures on General Equilibrium Theory
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Lectures on General Equilibrium Theory * * * The existence of equilibrium – p. 7/21
Exchange Economy

Define $\hat{x}^a : R_{++}^l \rightarrow R_+^l$ by $\hat{x}^a(p) = \bar{x}^a(p, p \cdot \omega^a)$. 
Exchange Economy

Define $\hat{x}^a : R^l_{++} \to R^l_+$ by $\hat{x}^a(p) = \bar{x}^a(p, p \cdot \omega^a)$.

Agent $a$’s excess demand function is $z^a(p) = \hat{x}^a(p) - \omega^a$. 
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Claim: $z^a$ is zero-homogeneous, i.e., $z^a(\lambda p) = z^a(p)$ for any scalar $\lambda > 0$, and $p \cdot z^a(p) = 0$ for all $p$. 

Exchange Economy
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Proof: $\bar{x}^a(p, w) = \text{argmax}_{x \in B(p, w)} U^a(x)$ is zero-homogeneous so

$$\hat{x}^a(\lambda p) = \bar{x}^a(\lambda p, (\lambda p) \cdot \omega^a) = \bar{x}^a(p, p \cdot \omega^a) = \hat{x}^a(p)$$

which in turn guarantees that $z^a(\lambda p) = z^a(p)$. 
Exchange Economy

Define \( \hat{x}^a : \mathbb{R}^{l+} \to \mathbb{R}^{l+} \) by \( \hat{x}^a(p) = \bar{x}^a(p, p \cdot \omega^a) \).

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Proof: \( \bar{x}^a(p, w) = \arg\max_{x \in B(p, w)} U^a(x) \) is zero-homogeneous so

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\hat{x}^a(\lambda p) = \bar{x}^a(\lambda p, (\lambda p) \cdot \omega^a) = \bar{x}^a(p, p \cdot \omega^a) = \hat{x}^a(p)
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which in turn guarantees that \( z^a(\lambda p) = z^a(p) \).

Since \( p \cdot \bar{x}^a(p, p \cdot \omega^a) = p \cdot \omega^a \), we have

\[
p \cdot [\bar{x}^a(p, p \cdot \omega^a) - \omega^a] = 0.
\]

So \( p \cdot z^a(p) = 0 \). QED
Exchange Economy

Aggregate (or market) demand at price $p$ is

$$X(p) = \sum_{a \in A} \hat{x}^a(p).$$

Aggregate excess demand function $Z : R^l_{++} \rightarrow R^l$ is

$$Z(p) = X(p) - \bar{\omega}.$$
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$Z$ is zero-homogeneous and obeys Walras’ Law, $p \cdot Z(p) = 0$ for all $p$. Both inherited from $z^a$, obviously.
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**Fundamental Question:** What conditions guarantee that there is $p^* \gg 0$ such that $Z(p^*) = 0$?
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Fundamental Question: What conditions guarantee that there is $p^* \gg 0$ such that $Z(p^*) = 0$?

Note that, since $Z$ is zero-homogeneous, if $p^*$ is an equilibrium price so is $\lambda p^*$ for any $\lambda > 0$. 

Lectures on General Equilibrium Theory * * * The existence of equilibrium – p. 9/21
Exchange Economy

With two agents A and B and two goods:

Agent A’s utility function $U^A(x_1, x_2) = \ln x_1 + 2 \ln x_2$. 
Exchange Economy

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Endowment $\omega^A = (1, 0)$, so $w^A = p_1$, and

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Lectures on General Equilibrium Theory ★ ★ ★ The existence of equilibrium – p. 10/21
Exchange Economy

With two agents A and B and two goods:

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Endowment $\omega^A = (1, 0)$, so $w^A = p_1$, and

$$\hat{x}^A(p) = \left( \frac{1}{3}, \frac{2p_1}{3p_2} \right).$$

Assume that agent B’s utility function is $U^B(x_1, x_2) = 2 \ln x_1 + \ln x_2$ and that his endowment is $(0, 1)$.
Exchange Economy

Exercise: show that

\[ Z(p) = \left( -\frac{2}{3} + \frac{2p_2}{3p_1}, \frac{2p_1}{3p_2} - \frac{2}{3} \right). \]

Setting \( Z_1(p) = 0 \) we obtain \( p_1 = p_2 \).
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Setting \( Z_2(p) = 0 \) we obtain \( p_1 = p_2 \).
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Equilibrium price is \((\lambda, \lambda)\) for any \( \lambda > 0 \).
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Note: if at \( p^* \), we have \( Z_1(p^*) = 0 \) then \( Z_2(p^*) = 0 \). This follows from Walras’ Law, which says that \( p_1^* Z_1(p^*) + p_2^* Z_2(p^*) = 0 \).
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Setting \( Z_2(p) = 0 \) we obtain \( p_1 = p_2 \).

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Note: if at \( p^* \), we have \( Z_1(p^*) = 0 \) then \( Z_2(p^*) = 0 \). This follows from Walras’ Law, which says that \( p_1^* Z_1(p^*) + p_2^* Z_2(p^*) = 0 \).
Clear from picture that the existence of a solution to \(Z(p) = 0\) requires the \textit{continuity} of \(Z\) and also the right \textit{boundary condition} ...

Continuity of \(Z\) is guaranteed if \(\bar{x}^a\) is continuous for all \(a\). This is in turn guaranteed by (P1) (the continuity of \(U^a\) for all agents \(a\)).

Recall the boundary property satisfied by \(\bar{x}_a\): if \((p^n, w^n) \rightarrow (\bar{p}, \bar{w})\) such that \(\bar{w} > 0\) and \(I = \{i : \bar{p}_i = 0\}\) is nonempty, then

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\sum_{i \in I} \bar{x}^a_i (p^n, w^n) \rightarrow \infty.
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$$\sum_{i \in I} \bar{x}_i^a (p^n, w^n) \to \infty.$$

Example: Cobb-Douglas demand for good $j$ is $\alpha_j \frac{w^n}{p_j}$. Clearly tends to infinity if $p_j \to 0$ and $w^n \to \bar{w}$, with $\bar{w} > 0$. 
Exchange Economy

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Example: Cobb-Douglas demand for good $j$ is $\alpha_j \frac{w^n}{p_j}$. Clearly tends to infinity if $p_j \rightarrow 0$ and $w^n \rightarrow \bar{w}$, with $\bar{w} > 0$.

The requirement that $\bar{w} > 0$ is crucial. Compare $p^n = (1, \frac{1}{n})$ with $\omega = (1, 0)$ and $\omega = (0, 1)$; $w^n$ tends to 1 in the first case and 0 in the second...
Corollary: In economy $\mathcal{E}$, suppose $p^n$ tends to $\bar{p} \neq 0$, such that $I = \{i : \bar{p}_i = 0\}$ is nonempty. Then $\sum_{i \in I} Z_i(p^n) \to \infty$. 
Exchange Economy

Corollary: In economy $E$, suppose $p^n$ tends to $\bar{p} \neq 0$, such that $I = \{i : p_i = 0\}$ is nonempty. Then $\sum_{i \in I} Z_i(p^n) \to \infty$.

Proof: Suppose that for good $k$, $\bar{p}_k > 0$. Since $\bar{\omega} = \sum_{a \in A} \omega^a \gg 0$, there is $\tilde{a}$ with $\omega^\tilde{a}_k > 0$. (In other words, there is some agent $\tilde{a}$ who has a strictly positive endowment of good $k$.) Then $p^n \cdot \omega^\tilde{a}$ tends to $\bar{\omega}^\tilde{a} = \bar{p} \cdot \omega^\tilde{a} > 0$. 
Corollary: In economy \( E \), suppose \( p^n \) tends to \( \bar{p} \neq 0 \), such that \( I = \{ i : \bar{p}_i = 0 \} \) is nonempty. Then \( \sum_{i \in I} Z_i(p^n) \to \infty \).

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So \( \sum_{i \in I} \hat{x}_{i}^{\tilde{a}}(p^n) = \sum_{i \in I} \bar{x}_{i}^{\tilde{a}}(p^n, p^n \cdot \omega^{\tilde{a}}) \to \infty \), which implies that

\[
\sum_{i \in I} Z_i(p^n) = \sum_{i \in I} X_i(p^n) - \sum_{i \in I} \bar{\omega}_i \\
\geq \sum_{i \in I} \bar{x}_{i}^{\tilde{a}}(p^n, p^n \cdot \omega^{\tilde{a}}) - \sum_{i \in I} \bar{\omega}_i \to \infty.
\]

QED
Exchange Economy

Theorem: The excess demand function $Z : R^l_+ \rightarrow R^l$ of the economy $E$ has the following properties:

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**Note:** Clear that $Z$ is bounded below since

$$Z(p) = X(p) - \bar{\omega} > -\bar{\omega}.$$
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Theorem (Arrow and Debreu; McKenzie): Suppose $Z$ satisfies properties (1) to (5). Then there is $p^*$ such that $Z(p^*) = 0$.

Proof uses Kakutani’s fixed point theorem, which generalizes Brouwer’s fixed point theorem to correspondences.

Brouwer’s fixed point theorem is a (far-reaching) generalization of the intermediate value theorem.
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Brouwer’s fixed point theorem is a (far-reaching) generalization of the intermediate value theorem.

**Intermediate value theorem** Let $f$ be a continuous function defined on some interval $[a, b]$. If $f(a)$ and $f(b)$ are of different signs, then there is $c \in [a, b]$ such that $f(c) = 0$. 
Exchange Economy

When the economy has just two goods, existence can be proved using the intermediate value theorem.

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Now consider \( p \rightarrow \infty \).

Then \( Z_1(1, p) = Z_1(\frac{1}{p}, 1) \rightarrow \infty \). In particular, there is \( p'' \) such that \( Z_1(1, p'') > 0 \). This implies that \( Z_2(1, p'') < 0 \) (by Walras’ Law).
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So there is $p'$ and $p''$ such that $Z_2(1, p') > 0$ and $Z_2(1, p'') < 0$. By IVT, there is $p^*$ such that $Z_2(1, p^*) = 0$. QED
Brouwer’s fixed point theorem

Intermediate value theorem can be re-stated as a fixed point theorem.

**Theorem** Let $K$ be a compact (i.e., closed and bounded) interval and suppose that $\phi : K \rightarrow K$ is a continuous function. Then there is $x^* \in K$ such that $\phi(x^*) = x^*$.
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Brouwer’s fixed point theorem Let $K$ be a compact and convex set in $\mathbb{R}^l$ and suppose that the function $\phi : K \rightarrow K$ is continuous. Then there is $x^*$ such that $\phi(x^*) = x^*$. 

Lectures on General Equilibrium Theory
Proof of equilibrium existence

We present a simple proof of equilibrium existence in the case where \( \omega_a \gg 0 \) for all \( a \).

Define \( \tilde{B}(p, a) = B(p, p \cdot \omega_a) \cap \{ x \leq 2\bar{\omega} \} \).
This is a truncated budget set for agent \( a \).
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Assuming (P1), (P2), and (P3), then \( \arg\max_{x \in \tilde{B}(p,a)} U^a(x) \) exists and is unique for all \( p \in \Delta = \{ p > 0 : \sum_{i=1}^l p_i = 1 \} \).
We denote this (modified demand) by \( \tilde{x}^a(p) \).
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The crucial feature of \( \tilde{x}^a \) that makes it useful is that it is also defined on the boundary of \( \Delta \) (the unit simplex), unlike \( \hat{x}^a \) which is defined only in the interior of \( \Delta \).
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The crucial feature of \( \tilde{x}^a \) that makes it useful is that it is also defined on the boundary of \( \Delta \) (the unit simplex), unlike \( \hat{x}^a \) which is defined only in the interior of \( \Delta \).

Note that \( \tilde{x}^a \) satisfies \( p \cdot \tilde{x}^a(p) = p \cdot \omega^a \). Furthermore, \( \tilde{x}^a \) is continuous in \( p \). (This property relies crucially on \( \omega^a \gg 0 \) – can you see why?)
Proof of equilibrium existence

Therefore, map $\tilde{Z} : \Delta \to \mathbb{R}^l$ defined by

$$\tilde{Z}(p) = \sum_{a \in A} [\tilde{x}^a(p) - \omega^a]$$

is continuous and obeys Walras’ Law.

**Lemma 1** There is $p^* \gg 0$ such that $\tilde{Z}(p^*) = 0$.

**Lemma 2** If there is $p^* \gg 0$ such that $\tilde{Z}(p^*) = 0$, then $Z(p^*) = 0$. 
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Proof of Lemma 1: Define \( \psi : \Delta \to \Delta \) by

\[
\psi_j(p) = \frac{p_j + \max\{\tilde{Z}_j(p), 0\}}{1 + \sum_{i=1}^l \max\{\tilde{Z}_i(p), 0\}} \quad \text{for all } j.
\]

\( \Delta \) is compact and convex set and this map is continuous. Brouwer’s theorem guarantees that there is \( p^* \) such that \( \psi(p^*) = p^* \).
Proof of equilibrium existence

Therefore, map $\tilde{Z} : \Delta \to \mathbb{R}^l$ defined by

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$\Delta$ is compact and convex set and this map is continuous. Brouwer’s theorem guarantees that there is $p^*$ such that $\psi(p^*) = p^*$.

If $p^*_k = 0$ for some $k$, then $\max\{\tilde{Z}_k(p^*), 0\} > 0$ and so $\psi_k(p^*) \neq p^*_k = 0$ – contradiction.
Proof of equilibrium existence

So \( p^* \gg 0 \). By Walras’ Law, there is \( h \) such that \( \max\{\tilde{Z}_h(p^*), 0\} = 0 \).

Since \( \psi(p^*) = p^* \), in particular,

\[
p_h^* = \psi_h(p^*) = \frac{p_h^* + 0}{1 + \sum_{i=1}^{l} \max\{\tilde{Z}_i(p), 0\}}.
\]

This gives

\[
\sum_{i=1}^{l} \max\{\tilde{Z}_i(p), 0\} = 0,
\]

so \( \tilde{Z}_i(p^*) \leq 0 \) for all \( i \). By Walras’ Law and the fact that \( p^* \gg 0 \), we have \( \tilde{Z}_i^*(p^*) = 0 \) for all \( i \).

QED
Proof of equilibrium existence

Lemma 2 If there is \( p^* \gg 0 \) such that \( \tilde{Z}(p^*) = 0 \), then \( Z(p^*) = 0 \).

Proof: We claim that \( \hat{x}^a(p^*) = \tilde{x}^a(p^*) \) for all agents. Clearly, this implies that \( Z(p^*) = \tilde{Z}(p^*) = 0 \).

Suppose, to the contrary, that for some agent \( b \), \( \hat{x}^b(p^*) \neq \tilde{x}^b(p^*) \), which means that \( \hat{x}^b(p^*) \) is not less than \( 2\bar{\omega} \) and \( U^b(\hat{x}^b(p^*)) > U^b(\tilde{x}^b(p^*)) \).

Since \( \tilde{Z}(p^*) = 0 \), it must be the case that \( \tilde{x}^b(p^*) \ll 2\bar{\omega} \).
Proof of equilibrium existence

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Since $\tilde{Z}(p^*) = 0$, it must be the case that $\tilde{x}^b(p^*) \ll 2\bar{\omega}$.

Choose $t \in (0, 1)$ such that $x = t\tilde{x}^b(p^*) + (1 - t)\hat{x}^b(p^*)$ satisfies $x \ll 2\bar{\omega}$.

Note that $U^b(x) > U^b(\tilde{x}^b(p^*))$ (by the strict quasiconcavity of $U^b$) and that $x \in \tilde{B}(p^*, b)$.

This contradicts the optimality of $\tilde{x}^b(p^*)$ in $\tilde{B}(p^*, b)$. QED