# *Lectures on General Equilibrium Theory*

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# *The existence of equilibrium*

John Quah

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# Notation

The set

$$R^l = \{x = (x_1, x_2, ..., x_l) : x_i \in R\}$$

is known as the Euclidean space. We denote

$$\begin{aligned} R^l_+ &= \{x \in R^l : x_i \geq 0 \,\forall\, i\} \\ R^l_{++} &= \{x \in R^l : x_i > 0 \,\forall\, i\} \end{aligned}$$

We sometimes refer to $R^l_+$ as the positive orthant.

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Finally, $x \gg y$ if $x_i > y_i$ for all $i$.

Examples: $(1, 3, 3) > (1, 2, 3)$ and $(1, 3, 3) \gg (0, 2, 1)$.

# Demand Theory

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$U$ is quasiconcave if, for any $\alpha$, the set $\{x \in R^l_+ : U(x) \geq \alpha\}$ is a convex set. Equivalently, whenever $U(x') \geq \alpha$ and $U(x) \geq \alpha$, then $U(tx' + (1-t)x) \geq \alpha$ for any $t \in [0, 1]$.

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At price (vector) $p = (p_1, p_2, ..., p_l)$ in $R^l_{++}$ (we also write $p \gg 0$) and income $w > 0$, the agent's budget set is

$$B(p, w) = \{x \in R^l_+ : p \cdot x \leq w\}.$$

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Note that $B(tp, tw) = B(p, w)$ for any $t > 0$, so

$$\text{argmax}_{x \in B(p,w)} U(x) = \text{argmax}_{x \in B(tp,tw)} U(x).$$

# Demand Theory

Let $x^* \in \text{argmax}_{x \in B(p,w)} U(x)$.

Quite easy to see that

(A) if $U$ is increasing, then $p \cdot x^* = w$. In this case, we say that the agent's demand obeys the budget identity.

(B) if $U$ is strictly quasiconcave, then $\text{argmax}_{x \in B(p,w)} U(x)$ has *at most* one element.

(C) $\text{argmax}_{x \in B(p,w)} U(x)$ is nonempty if $U$ is a continuous function. (This is a straightforward consequence of Weierstrass Theorem.[a])

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[a] **Weierstrass Theorem**: *Suppose that* $K$ *is a compact set in* $R^l$ *and* $F : K \to R$ *a continuous function. Then* $\arg\max_{x \in K} F(x)$ *is nonempty.*

# Demand Theory

Result below summarizes (A)-(C) and more.

Proposition: Suppose that the utility function $U : R^l_+ \to R$ is (P1) continuous, (P2) strictly increasing, and (P3) strictly quasiconcave.

Then for any $(p, w)$ in $R^l_{++} \times R_{++}$, there exists a *unique* element $x^*$ in argmax$_{x \in B(p,w)} U(x)$.

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The function $\bar{x} : R^l_{++} \times R_{++} \to R^l_+$ mapping $(p, w)$ to $\bar{x}(p, w) =$ argmax$_{x \in B(p,w)} U(x)$ has the following properties:

(a) it is continuous,
(b) it obeys the budget identity [i.e., $p \cdot \bar{x}(p, w) = w$],
(c) it is zero-homogeneous, [i.e. $\bar{x}(tp, tw) = \bar{x}(p, w)$ for any $t > 0$] and
(d) it obeys this boundary condition: if $(p^n, w^n) \to (\bar{p}, \bar{w})$ such that $\bar{w} > 0$ and $I = \{i : \bar{p}_i = 0\}$ is nonempty, then

$$\sum_{i \in I} \bar{x}^a_i(p^n, w^n) \to \infty.$$

# Exchange Economy

Assume that there is a finite set $A$ of agents.

Agent $a \in A$ has the utility function $U^a : R^l_+ \to R$ and endowment $\omega^a = (\omega^a_1, \omega^a_2, ..., \omega^a_l)$ in $R^l_+$.

We require $U^a$ to obey (P1), (P2), and (P3), and that aggregate endowment

$$\bar{\omega} = \sum_{a \in A} \omega^a \gg 0.$$

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Claim: $z^a$ is zero-homogeneous, i.e., $z^a(\lambda p) = z^a(p)$ for any scalar $\lambda > 0$, and $p \cdot z^a(p) = 0$ for all $p$.

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Proof: $\bar{x}^a(p, w) = \text{argmax}_{x \in B(p,w)} U^a(x)$ is zero-homogeneous so

$$\hat{x}^a(\lambda p) = \bar{x}^a(\lambda p, (\lambda p) \cdot \omega^a) = \bar{x}^a(p, p \cdot \omega^a) = \hat{x}^a(p)$$

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Since $p \cdot \bar{x}^a(p, p \cdot \omega^a) = p \cdot \omega^a$, we have

$$p \cdot [\bar{x}^a(p, p \cdot \omega^a) - \omega^a] = 0.$$

So $p \cdot z^a(p) = 0$. **QED**

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$$X(p) = \sum_{a \in A} \hat{x}^a(p).$$

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Fundamental Question: What conditions guarantee that there is $p^* \gg 0$ such that $Z(p^*) = 0$?

Note that, since $Z$ is zero-homogeneous, if $p^*$ is an equilibrium price so is $\lambda p^*$ for any $\lambda > 0$.

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Agent A's utility function $U^A(x_1, x_2) = \ln x_1 + 2 \ln x_2$.

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Endowment $\omega^A = (1, 0)$, so $w^A = p_1$, and

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Assume that agent B's utility function is $U^B(x_1, x_2) = 2\ln x_1 + \ln x_2$ and that his endowment is $(0, 1)$.

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Exercise: show that

$$Z(p) = \left( -\frac{2}{3} + \frac{2p_2}{3p_1}, \frac{2p_1}{3p_2} - \frac{2}{3} \right).$$

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Note: if at $p^*$, we have $Z_1(p^*) = 0$ then $Z_2(p^*) = 0$. This follows from Walras' Law, which says that $p_1^* \, Z_1(p^*) + p_2^* \, Z_2(p^*) = 0$.

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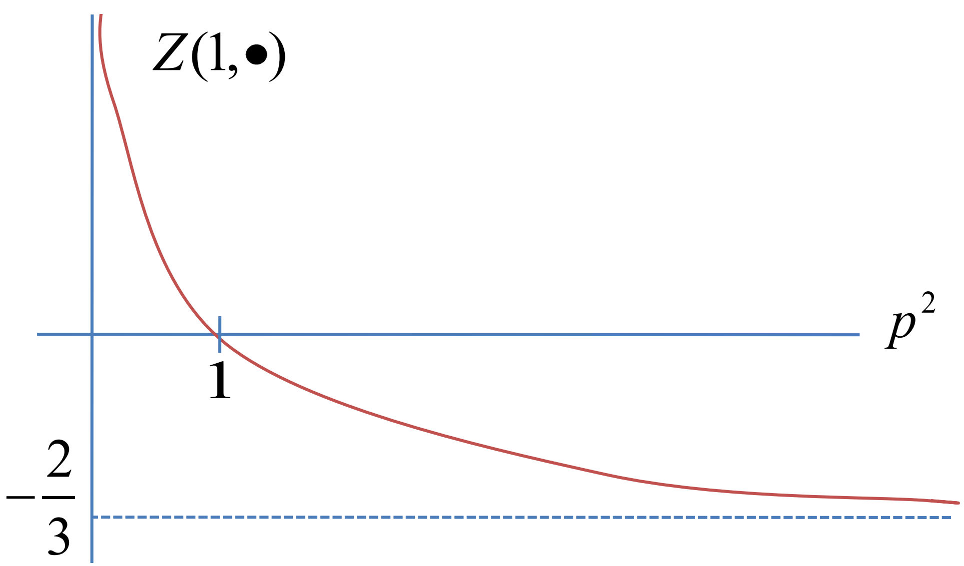

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Clear from picture that the existence of a solution to $Z(p) = 0$ requires the continuity of $Z$ and also the right boundary condition ...

Continuity of $Z$ is guaranteed if $\bar{x}^a$ is continuous for all $a$. This is in turn guaranteed by (P1) (the continuity of $U^a$ for all agents $a$).

Recall the boundary property satisfied by $\bar{x}_a$: if $(p^n, w^n) \to (\bar{p}, \bar{w})$ such that $\bar{w} > 0$ and $I = \{i : \bar{p}_i = 0\}$ is nonempty, then $\sum_{i \in I} \bar{x}_i^a(p^n, w^n) \to \infty$.

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The requirement that $\bar{w} > 0$ is crucial. Compare $p^n = (1, \frac{1}{n})$ with $\omega = (1, 0)$ and $\omega = (0, 1)$; $w^n$ tends to 1 in the first case and 0 in the second...

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Corollary: In economy $\mathcal{E}$, suppose $p^n$ tends to $\bar{p} \neq 0$, such that $I = \{i : \bar{p}_i = 0\}$ is nonempty. Then $\sum_{i \in I} Z_i(p^n) \to \infty$.

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Proof: Suppose that for good $k$, $\bar{p}_k > 0$. Since $\bar{\omega} = \sum_{a \in A} \omega^a \gg 0$, there is $\tilde{a}$ with $\omega_k^{\tilde{a}} > 0$. (In other words, there is some agent $\tilde{a}$ who has a strictly positive endowment of good $k$.) Then $p^n \cdot \omega^{\tilde{a}}$ tends to $\bar{w}^{\tilde{a}} = \bar{p} \cdot \omega^{\tilde{a}} > 0$.

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So $\sum_{i \in I} \hat{x}_i^{\tilde{a}}(p^n) = \sum_{i \in I} \bar{x}_i^{\tilde{a}}(p^n, p^n \cdot \omega^{\tilde{a}}) \to \infty$, which implies that

$$\begin{aligned}
\sum_{i \in I} Z_i(p^n) &= \sum_{i \in I} X_i(p^n) - \sum_{i \in I} \bar{\omega}_i \\
&\geq \sum_{i \in I} \bar{x}_i^{\tilde{a}}(p^n, p^n \cdot \omega^{\tilde{a}}) - \sum_{i \in I} \bar{\omega}_i \to \infty.
\end{aligned}$$

**QED**

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Brouwer's fixed point theorem is a (far-reaching) generalization of the intermediate value theorem.

Intermediate value theorem Let $f$ be a continuous function defined on some interval $[a, b]$. If $f(a)$ and $f(b)$ are of different signs, then there is $c \in [a, b]$ such that $f(c) = 0$.

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Then $Z_1(1, p) = Z_1(\frac{1}{p}, 1) \to \infty$. In particular, there is $p''$ such that $Z_1(1, p'') > 0$. This implies that $Z_2(1, p'') < 0$ (by Walras' Law).

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So there is $p'$ and $p''$ such that $Z_2(1, p') > 0$ and $Z_2(1, p'') < 0$.By IVT, there is $p^*$ such that $Z_2(1, p^*) = 0$. **QED**

# Brouwer's fixed point theorem

Intermediate value theorem can be re-stated as a fixed point theorem.

Theorem Let $K$ be a compact (i.e., closed and bounded) interval and suppose that $\phi : K \to K$ is a continous function. Then there is $x^* \in K$ such that $\phi(x^*) = x^*$.

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Brouwer's fixed point theorem Let $K$ be a compact and convex set in $R^l$ and suppose that the function $\phi : K \rightarrow K$ is continuous. Then there is $x^*$ such that $\phi(x^*) = x^*$.

# Proof of equilibrium existence

We present a simple proof of equilibrium existence in the case where $\omega_a \gg 0$ for all $a$.

Define $\tilde{B}(p,a) = B(p, p \cdot \omega_a) \cap \{x \leq 2\bar{\omega}\}$.
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Assuming (P1), (P2), and (P3), then $\text{argmax}_{x \in \tilde{B}(p,a)} U^a(x)$ exists and is unique for all $p \in \Delta = \{p > 0 : \sum_{i=1}^{l} p_i = 1\}$.
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Note that $\tilde{x}^a$ satisfies $p \cdot \tilde{x}^a(p) = p \cdot \omega^a$. Furthermore, $\tilde{x}^a$ is continuous in $p$. (This property relies crucially on $\omega^a \gg 0$ – can you see why?)

# Proof of equilibrium existence

Therefore, map $\tilde{Z} : \Delta \to R^l$ defined by

$$\tilde{Z}(p) = \sum_{a \in A} [\tilde{x}^a(p) - \omega^a]$$

is continuous and obeys Walras' Law.

Lemma 1 There is $p^* \gg 0$ such that $\tilde{Z}(p^*) = 0$.

Lemma 2 If there is $p^* \gg 0$ such that $\tilde{Z}(p^*) = 0$, then $Z(p^*) = 0$.

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Proof of Lemma 1: Define $\psi : \Delta \to \Delta$ by

$$\psi_j(p) = \frac{p_j + \max\{\tilde{Z}_j(p), 0\}}{1 + \sum_{i=1}^{l} \max\{\tilde{Z}_i(p), 0\}} \quad \text{for all } j.$$

$\Delta$ is compact and convex set and this map is continuous. Brouwer's theorem guarantees that there is $p^*$ such that $\psi(p^*) = p^*$.

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If $p_k^* = 0$ for some $k$, then $\max\{\tilde{Z}_k(p^*), 0\} > 0$ and so $\psi_k(p^*) \neq p_k^* = 0$ – contradiction.

# Proof of equilibrium existence

So $p^* \gg 0$. By Walras' Law, there is $h$ such that $\max\{\tilde{Z}_h(p^*), 0\} = 0$. Since $\psi(p^*) = p^*$, in particular,

$$p_h^* = \psi_h(p^*) = \frac{p_h^* + 0}{1 + \sum_{i=1}^{l} \max\{\tilde{Z}_i(p), 0\}}.$$

This gives

$$\sum_{i=1}^{l} \max\{\tilde{Z}_i(p), 0\} = 0,$$

so $\tilde{Z}_i(p^*) \leq 0$ for all $i$. By Walras' Law and the fact that $p^* \gg 0$, we have $\tilde{Z}_i^*(p^*) = 0$ for all $i$. QED

# Proof of equilibrium existence

Lemma 2 If there is $p^* \gg 0$ such that $\tilde{Z}(p^*) = 0$, then $Z(p^*) = 0$.

Proof: We claim that $\hat{x}^a(p^*) = \tilde{x}^a(p^*)$ for all agents. Clearly, this implies that $Z(p^*) = \tilde{Z}(p^*) = 0$.

Suppose, to the contrary, that for some agent $b$, $\hat{x}^b(p^*) \neq \tilde{x}^b(p^*)$, which means that $\hat{x}^b(p^*)$ is not less than $2\bar{\omega}$ and $U^b(\hat{x}^b(p^*)) > U^b(\tilde{x}^b(p^*))$.

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Choose $t \in (0,1)$ such that $x = t\tilde{x}^b(p^*) + (1-t)\hat{x}^b(p^*)$ satisfies $x \ll 2\bar{\omega}$.

Note that $U^b(x) > U^b(\tilde{x}^b(p^*))$ (by the strict quasiconcavity of $U^b$) and that $x \in \tilde{B}(p^*, b)$.

This contradicts the optimality of $\tilde{x}^b(p^*)$ in $\tilde{B}(p^*, b)$. QED