Lectures on General Equilibrium Theory Michaelmas 2008

John Quah

⋆ ⋆

john.quah@economics.ox.ac.uk

Readings on General Equilibrium Theory

The most relevant book for these lectures is Hal Varian's *Microeconomic Analysis* (chapters 17, 18, and 21). This book is an advanced version of his *Intermediate Microeconomics*, which most of you are familiar with.

When we get to the material on incomplete markets, you should also read Chapter 8 in Timothy van Zandt's draft book, *Introduction to the Economics of Uncertainty and Information*. You can find it at

http://faculty.insead.edu/vanzandt/327files/mybook.html

The treatment given in *Microeconomic Theory* (chapters 15, 16, 17, and 19) by Mas-Colell, Whinston and Green, is more comprehensive and thus useful, but it is also more demanding.

Notation

The set

$$R^l = \{x = (x_1, x_2, ..., x_l) : x_i \in R\}$$

is known as the Euclidean space. We denote

$$\begin{aligned} R^l_+ &= \{x \in R^l : x_i \geq 0 \, \forall \, i\} \\ R^l_{++} &= \{x \in R^l : x_i > 0 \, \forall \, i\} \end{aligned}$$

We sometimes refer to R^l_+ as the positive orthant.

For vectors x and y in R^l, we say

$x \geq y$ if $x_i \geq y_i$ for all i and

Notation

The set

$$R^l = \{x = (x_1, x_2, ..., x_l) : x_i \in R\}$$

is known as the Euclidean space. We denote

$$\begin{aligned} R^l_+ &= \{x \in R^l : x_i \geq 0 \, \forall \, i\} \\ R^l_{++} &= \{x \in R^l : x_i > 0 \, \forall \, i\} \end{aligned}$$

We sometimes refer to R^l_+ as the positive orthant.

For vectors x and y in R^l, we say

$x \geq y$ if $x_i \geq y_i$ for all i and $x > y$ if $x_i \geq y_i$ for all i and $x \neq y$.

Notation

The set

$$R^l = \{x = (x_1, x_2, ..., x_l) : x_i \in R\}$$

is known as the Euclidean space. We denote

$$\begin{aligned} R^l_+ &= \{x \in R^l : x_i \geq 0 \,\forall\, i\} \\ R^l_{++} &= \{x \in R^l : x_i > 0 \,\forall\, i\} \end{aligned}$$

We sometimes refer to R^l_+ as the positive orthant.

For vectors x and y in R^l, we say

$x \geq y$ if $x_i \geq y_i$ for all i and $x > y$ if $x_i \geq y_i$ for all i and $x \neq y$.

Finally, $x \gg y$ if $x_i > y_i$ for all i.

Notation

The set

$$R^l = \{x = (x_1, x_2, ..., x_l) : x_i \in R\}$$

is known as the Euclidean space. We denote

$$\begin{aligned} R^l_+ &= \{x \in R^l : x_i \geq 0 \,\forall\, i\} \\ R^l_{++} &= \{x \in R^l : x_i > 0 \,\forall\, i\} \end{aligned}$$

We sometimes refer to R^l_+ as the positive orthant.

For vectors x and y in R^l, we say

$x \geq y$ if $x_i \geq y_i$ for all i and $x > y$ if $x_i \geq y_i$ for all i and $x \neq y$.

Finally, $x \gg y$ if $x_i > y_i$ for all i.

Examples: $(1, 3, 3) > (1, 2, 3)$ and $(1, 3, 3) \gg (0, 2, 1)$.

Demand Theory

Assume that there are l commodities and that the consumption space is R^l_+ (the positive orthant).

Agent has the utility function $U : R^l_+ \to R$.

Demand Theory

Assume that there are l commodities and that the consumption space is R^l_+ (the positive orthant).

Agent has the utility function $U : R^l_+ \to R$.

At price (vector) $p = (p_1, p_2, ..., p_l)$ in R^l_{++} (we also write $p \gg 0$) and income $w > 0$, the budget set is

$$B(p, w) = \{x \in R^l_+ : p \cdot x \le w\}.$$

Demand Theory

Assume that there are l commodities and that the consumption space is R^l_+ (the positive orthant).

Agent has the utility function $U : R^l_+ \to R$.

At price (vector) $p = (p_1, p_2, ..., p_l)$ in R^l_{++} (we also write $p \gg 0$) and income $w > 0$, the budget set is

$$B(p, w) = \{x \in R^l_+ : p \cdot x \leq w\}.$$

A commodity bundle x^* is the demand of the agent at (p, w) if x^* maximizes utility in $B(p, w)$; formally,

$$x^* \in \text{argmax}_{x \in B(p,w)} U(x).$$

Demand Theory

Assume that there are l commodities and that the consumption space is R^l_+ (the positive orthant).

Agent has the utility function $U : R^l_+ \to R$.

At price (vector) $p = (p_1, p_2, ..., p_l)$ in R^l_{++} (we also write $p \gg 0$) and income $w > 0$, the budget set is

$$B(p, w) = \{x \in R^l_+ : p \cdot x \leq w\}.$$

A commodity bundle x^* is the demand of the agent at (p, w) if x^* maximizes utility in $B(p, w)$; formally,

$$x^* \in \text{argmax}_{x \in B(p,w)} U(x).$$

What condition on U is sufficient to guarantee that x^* exists?

Demand Theory

Assume that there are l commodities and that the consumption space is R^l_+ (the positive orthant).

Agent has the utility function $U : R^l_+ \rightarrow R$.

At price (vector) $p = (p_1, p_2, ..., p_l)$ in R^l_{++} (we also write $p \gg 0$) and income $w > 0$, the budget set is

$$B(p, w) = \{x \in R^l_+ : p \cdot x \leq w\}.$$

A commodity bundle x^* is the demand of the agent at (p, w) if x^* maximizes utility in $B(p, w)$; formally,

$$x^* \in \text{argmax}_{x \in B(p,w)} U(x).$$

What condition on U is sufficient to guarantee that x^* exists?

Answer: (P1) When U is a continuous function.

Demand Theory

A set K in R^l is said to be bounded if there is $B > 0$ such that for any x in K, $|x_i| < B$.

K is said to be closed if whenever the sequence x^n is in K and x^n converges to $\bar{x}$, then $\bar{x}$ is also in K.

Loosely speaking, a closed set is one that contains its boundary.

Any set that is closed and bounded is said to be compact.

Demand Theory

A set K in R^l is said to be bounded if there is $B > 0$ such that for any x in K, $|x_i| < B$.

K is said to be closed if whenever the sequence x^n is in K and x^n converges to $\bar{x}$, then $\bar{x}$ is also in K.

Loosely speaking, a closed set is one that contains its boundary.

Any set that is closed and bounded is said to be compact.

Weierstrass Theorem: Suppose that K is a compact set in R^l and $F : K \to R$ a continuous function. Then

$$\text{argmax}_{x \in K} F(x) \neq \emptyset.$$

Demand Theory

Proposition: Suppose $U : R^l_+ \rightarrow R$ is a continuous function. Then for any $p \gg 0$ and $w > 0$, the demand set $\text{argmax}_{x \in B(p,w)} U(x)$ is nonempty.

Proof: The budget set $B(p, w)$ is a compact set. So result follows immediately from Weierstrass Theorem. **QED**

Demand Theory

Question: Suppose $x^* \in \text{argmax}_{x \in B(p,w)} U(x)$.
When does x^* obey the budget identity, i.e., $p \cdot x^* = w$?

Demand Theory

Question: Suppose $x^* \in$ argmax$_{x \in B(p,w)} U(x)$.
When does x^* obey the budget identity, i.e., $p \cdot x^* = w$?

Answer: (P2) When U is strictly monotone, i.e.,
whenever $x'' > x'$, we have $U(x'') > U(x')$.

Proof: Suppose, to the contrary, that $p \cdot x^* < w$. Then there is $\epsilon > 0$ and sufficiently small such that $x^{**} = x^* + (\epsilon, \epsilon, ..., \epsilon)$ is in $B(p, w)$, i.e.,
$p \cdot x^{**} = p \cdot x^* + p \cdot (\epsilon, \epsilon, ..., \epsilon) < w$.

By monotonicity, $U(x^{**}) > U(x^*)$ - contradiction. **QED**

Demand Theory

Question: When is $\text{argmax}_{x \in B(p,w)} U(x)$ unique?

Demand Theory

Question: When is $\text{argmax}_{x \in B(p,w)} U(x)$ unique?

Answer: (P3) When U is strictly quasi-concave, i.e.,

if $x' \neq x''$ and $U(x') = U(x'')$, then for any $\lambda \in (0,1)$

$$U(\lambda x' + (1-\lambda)x'') > U(x') = U(x'').$$

Proof: Suppose to the contrary that x^* and x^{**} are both in $\text{argmax}_{x \in B(p,w)} U(x)$. Then $U(x^*) = U(x^{**})$.

The bundle $x' = 0.5x^* + 0.5x^{**}$ is also in $B(p,w)$ (since $p \cdot x' \leq w$) and has a higher utility than x^* (or x^{**}). **QED**

Demand Theory

Question: When is $\text{argmax}_{x \in B(p,w)} U(x)$ unique?

Answer: (P3) When U is strictly quasi-concave, i.e.,

if $x' \neq x''$ and $U(x') = U(x'')$, then for any $\lambda \in (0,1)$

$$U(\lambda x' + (1-\lambda)x'') > U(x') = U(x'').$$

Proof: Suppose to the contrary that x^* and x^{**} are both in $\text{argmax}_{x \in B(p,w)} U(x)$. Then $U(x^*) = U(x^{**})$.

The bundle $x' = 0.5x^* + 0.5x^{**}$ is also in $B(p,w)$ (since $p \cdot x' \leq w$) and has a higher utility than x^* (or x^{**}). **QED**

Note: A sufficient condition for U to be strictly quasi-concave is for it be strictly concave, i.e., for $x' \neq x''$ and t in $(0,1)$, we have

$$U(tx' + (1-t)x'') > tU(x') + (1-t)U(x'').$$

Check this yourself; check also that quasi-concavity (but not concavity) is an ordinal property.

Demand Theory

To recap: if U obeys (P1) and (P3), then for any (p, w) in $R^l_{++} \times R_{++}$, there exists a *unique* element x^* in argmax$_{x \in B(p,w)} U(x)$.

So we can define a function $\bar{x} : R^l_{++} \times R_{++} \rightarrow R^l_{+}$ mapping

(p, w) to $\bar{x}(p, w) =$ argmax$_{x \in B(p,w)} U(x)$.

We refer to this function as the demand function generated by U.

Demand Theory

To recap: if U obeys (P1) and (P3), then for any (p, w) in $R^l_{++} \times R_{++}$, there exists a *unique* element x^* in $\text{argmax}_{x \in B(p,w)} U(x)$.

So we can define a function $\bar{x} : R^l_{++} \times R_{++} \to R^l_{+}$ mapping

(p, w) to $\bar{x}(p, w) = \text{argmax}_{x \in B(p,w)} U(x)$.

We refer to this function as the demand function generated by U.

(P1) also guarantees that $\bar{x}$ is a *continuous* function.

Furthermore, if U obeys (P2) then $\bar{x}$ obeys the budget identity, i.e., $p \cdot \bar{x}(p, w) = w$.

Demand Theory

Example: The Cobb-Douglas utility function

$$U(x) = \sum_{i=1}^{l} \alpha_i \ln x_i,$$

with $\sum_i^l \alpha_i = 1$. This satisfies (P1), (P2), and (P3) because the natural log function is continuous, increasing, and strictly concave (so, in fact, U is strictly concave).

The demand function generated by U is

$$\bar{x}(p, w) = \left(\alpha_1 \frac{w}{p_1}, \alpha_2 \frac{w}{p_2}, ..., \alpha_l \frac{w}{p_l} \right).$$

Note that the share of expenditure on good i is always α_i; in other words, when income is w, the agent always spends $\alpha_i w$ on good i, irrespective of the prices.

Exchange Economy

Assume that there is a finite set A of agents.

Agent $a \in A$ has the utility function $U^a : R^l_+ \to R$ and endowment $\omega^a = (\omega^a_1, \omega^a_2, ..., \omega^a_l)$ in R^l_+.

We require U^a to obey (P1), (P2), and (P3), and that aggregate endowment

$$\bar{\omega} = \sum_{a \in A} \omega^a \gg 0.$$

Exchange Economy

Assume that there is a finite set A of agents.

Agent $a \in A$ has the utility function $U^a : R^l_+ \rightarrow R$ and endowment $\omega^a = (\omega^a_1, \omega^a_2, ..., \omega^a_l)$ in R^l_+.

We require U^a to obey (P1), (P2), and (P3), and that aggregate endowment

$$\bar{\omega} = \sum_{a \in A} \omega^a \gg 0.$$

Agent a's demand function is $\bar{x}^a$.

Exchange Economy

Assume that there is a finite set A of agents.

Agent $a \in A$ has the utility function $U^a : R^l_+ \to R$ and endowment $\omega^a = (\omega^a_1, \omega^a_2, ..., \omega^a_l)$ in R^l_+.

We require U^a to obey (P1), (P2), and (P3), and that aggregate endowment

$$\bar{\omega} = \sum_{a \in A} \omega^a \gg 0.$$

Agent a's demand function is $\bar{x}^a$.

In an exchange economy, income is determined by the prevailing price p. With an endowment of ω^a, agent a's income $w^a = p \cdot \omega^a$.

Exchange Economy

Assume that there is a finite set A of agents.

Agent $a \in A$ has the utility function $U^a : R^l_+ \to R$ and endowment $\omega^a = (\omega^a_1, \omega^a_2, ..., \omega^a_l)$ in R^l_+.

We require U^a to obey (P1), (P2), and (P3), and that aggregate endowment

$$\bar{\omega} = \sum_{a \in A} \omega^a \gg 0.$$

Agent a's demand function is $\bar{x}^a$.

In an exchange economy, income is determined by the prevailing price p. With an endowment of ω^a, agent a's income $w^a = p \cdot \omega^a$.

His demand at price p is $\bar{x}^a(p, p \cdot \omega^a)$.

Exchange Economy

Assume that there is a finite set A of agents.

Agent $a \in A$ has the utility function $U^a : R^l_+ \to R$ and endowment $\omega^a = (\omega^a_1, \omega^a_2, ..., \omega^a_l)$ in R^l_+.

We require U^a to obey (P1), (P2), and (P3), and that aggregate endowment

$$\bar{\omega} = \sum_{a \in A} \omega^a \gg 0.$$

Agent a's demand function is $\bar{x}^a$.

In an exchange economy, income is determined by the prevailing price p. With an endowment of ω^a, agent a's income $w^a = p \cdot \omega^a$.

His demand at price p is $\bar{x}^a(p, p \cdot \omega^a)$.

Exchange Economy

Define $\hat{x}^{\,a} : R^l_{++} \to R^l_+$ by $\hat{x}^{\,a}(p) = \bar{x}^a(p, p \cdot \omega^a)$.

Exchange Economy

Define $\hat{x}^{\,a} : R^l_{++} \to R^l_+$ by $\hat{x}^{\,a}(p) = \bar{x}^a(p, p \cdot \omega^a)$.

Agent a's excess demand function is $z^a(p) = \hat{x}^a(p) - \omega^a$.

Exchange Economy

Define $\hat{x}^{\,a} : R^l_{++} \to R^l_+$ by $\hat{x}^{\,a}(p) = \bar{x}^a(p, p \cdot \omega^a)$.

Agent a's excess demand function is $z^a(p) = \hat{x}^a(p) - \omega^a$.

Claim: z^a is zero-homogeneous, i.e., $z^a(\lambda p) = z^a(p)$ for any scalar $\lambda > 0$, and $p \cdot z^a(p) = 0$ for all p.

Exchange Economy

Define $\hat{x}^{\,a} : R^l_{++} \to R^l_+$ by $\hat{x}^{\,a}(p) = \bar{x}^a(p, p \cdot \omega^a)$.

Agent a's excess demand function is $z^a(p) = \hat{x}^a(p) - \omega^a$.

Claim: z^a is zero-homogeneous, i.e., $z^a(\lambda p) = z^a(p)$ for any scalar $\lambda > 0$, and $p \cdot z^a(p) = 0$ for all p.

Proof: $\bar{x}^a(p, w) = \text{argmax}_{x \in B(p,w)} U^a(x)$ is zero-homogeneous since

Exchange Economy

Define $\hat{x}^{\,a} : R^l_{++} \to R^l_+$ by $\hat{x}^{\,a}(p) = \bar{x}^a(p, p \cdot \omega^a)$.

Agent a's excess demand function is $z^a(p) = \hat{x}^a(p) - \omega^a$.

Claim: z^a is zero-homogeneous, i.e., $z^a(\lambda p) = z^a(p)$ for any scalar $\lambda > 0$, and $p \cdot z^a(p) = 0$ for all p.

Proof: $\bar{x}^a(p, w) =$ argmax$_{x \in B(p,w)} U^a(x)$ is zero-homogeneous since $B(p, w) = B(\lambda p, \lambda w)$ for any $\lambda > 0$. So

$$\hat{x}^a(\lambda p) = \bar{x}^a(\lambda p, (\lambda p) \cdot \omega^a) = \bar{x}^a(p, p \cdot \omega^a) = \hat{x}^a(p)$$

which in turn guarantees that $z^a(\lambda p) = z^a(p)$.

Exchange Economy

Define $\hat{x}^{\,a} : R^l_{++} \to R^l_+$ by $\hat{x}^{\,a}(p) = \bar{x}^a(p, p \cdot \omega^a)$.

Agent a's excess demand function is $z^a(p) = \hat{x}^a(p) - \omega^a$.

Claim: z^a is zero-homogeneous, i.e., $z^a(\lambda p) = z^a(p)$ for any scalar $\lambda > 0$, and $p \cdot z^a(p) = 0$ for all p.

Proof: $\bar{x}^a(p, w) =$ argmax$_{x \in B(p,w)} U^a(x)$ is zero-homogeneous since $B(p, w) = B(\lambda p, \lambda w)$ for any $\lambda > 0$. So

$$\hat{x}^a(\lambda p) = \bar{x}^a(\lambda p, (\lambda p) \cdot \omega^a) = \bar{x}^a(p, p \cdot \omega^a) = \hat{x}^a(p)$$

which in turn guarantees that $z^a(\lambda p) = z^a(p)$.

Since $p \cdot \bar{x}^a(p, p \cdot \omega^a) = p \cdot \omega^a$, we have

$$p \cdot [\bar{x}^a(p, p \cdot \omega^a) - \omega^a] = 0.$$

So $p \cdot z^a(p) = 0$. **QED**

Exchange Economy

Aggregate (or market) demand at price p is

$$X(p) = \sum_{a \in A} \hat{x}^a(p).$$

Aggregate excess demand function $Z : R^l_{++} \to R^l$ is

$$Z(p) = X(p) - \bar{\omega}.$$

Exchange Economy

Aggregate (or market) demand at price p is

$$X(p) = \sum_{a \in A} \hat{x}^a(p).$$

Aggregate excess demand function $Z : R^l_{++} \to R^l$ is

$$Z(p) = X(p) - \bar{\omega}.$$

Z is zero-homogeneous and obeys Walras' Law, $p \cdot Z(p) = 0$ for all p. Both inherited from z^a, obviously.

Exchange Economy

Aggregate (or market) demand at price p is

$$X(p) = \sum_{a \in A} \hat{x}^a(p).$$

Aggregate excess demand function $Z : R^l_{++} \to R^l$ is

$$Z(p) = X(p) - \bar{\omega}.$$

Z is zero-homogeneous and obeys Walras' Law, $p \cdot Z(p) = 0$ for all p. Both inherited from z^a, obviously.

Fundamental Question: What conditions guarantee that there is $p^* \gg 0$ such that $Z(p^*) = 0$?

Exchange Economy

Aggregate (or market) demand at price p is

$$X(p) = \sum_{a \in A} \hat{x}^a(p).$$

Aggregate excess demand function $Z : R^l_{++} \to R^l$ is

$$Z(p) = X(p) - \bar{\omega}.$$

Z is zero-homogeneous and obeys Walras' Law, $p \cdot Z(p) = 0$ for all p. Both inherited from z^a, obviously.

Fundamental Question: What conditions guarantee that there is $p^* \gg 0$ such that $Z(p^*) = 0$?

Note that, since Z is zero-homogeneous, if p^* is an equilibrium price so is λp^* for any $\lambda > 0$.

Exchange Economy

With two agents A and B and two goods:

Agent A's utility function $U^A(x_1, x_2) = \ln x_1 + 2 \ln x_2$.

Exchange Economy

With two agents A and B and two goods:

Agent A's utility function $U^A(x_1, x_2) = \ln x_1 + 2 \ln x_2$.

Demand function $\bar{x}^A(p, w) = \left(\frac{w}{3p_1}, \frac{2w}{3p_2}\right)$.

Exchange Economy

With two agents A and B and two goods:

Agent A's utility function $U^A(x_1, x_2) = \ln x_1 + 2\ln x_2$.

Demand function $\bar{x}^A(p, w) = \left(\frac{w}{3p_1}, \frac{2w}{3p_2}\right)$.

Endowment $\omega^A = (1, 0)$, so $w^A = p_1$, and

$$\hat{x}^A(p) = \left(\frac{1}{3}, \frac{2p_1}{3p_2}\right).$$

Exchange Economy

With two agents A and B and two goods:

Agent A's utility function $U^A(x_1, x_2) = \ln x_1 + 2 \ln x_2$.

Demand function $\bar{x}^A(p, w) = \left(\frac{w}{3p_1}, \frac{2w}{3p_2} \right)$.

Endowment $\omega^A = (1, 0)$, so $w^A = p_1$, and

$$\hat{x}^A(p) = \left(\frac{1}{3}, \frac{2p_1}{3p_2} \right).$$

Assume that agent B's utility function is $U^B(x_1, x_2) = 2 \ln x_1 + \ln x_2$ and that his endowment is $(0, 1)$.

Exchange Economy

Exercise: show that

$$Z(p) = \left(-\frac{2}{3} + \frac{2p_2}{3p_1}, \frac{2p_1}{3p_2} - \frac{2}{3}\right).$$

Setting $Z_1(p) = 0$ we obtain $p_1 = p_2$.

Exchange Economy

Exercise: show that

$$Z(p) = \left(-\frac{2}{3} + \frac{2p_2}{3p_1}, \frac{2p_1}{3p_2} - \frac{2}{3} \right).$$

Setting $Z_1(p) = 0$ we obtain $p_1 = p_2$.

Setting $Z_2(p) = 0$ we obtain $p_1 = p_2$.

Exchange Economy

Exercise: show that

$$Z(p) = \left(-\frac{2}{3} + \frac{2p_2}{3p_1}, \frac{2p_1}{3p_2} - \frac{2}{3}\right).$$

Setting $Z_1(p) = 0$ we obtain $p_1 = p_2$.
Setting $Z_2(p) = 0$ we obtain $p_1 = p_2$.

Equilibrium price is (λ, λ) for any $\lambda > 0$.

Exchange Economy

Exercise: show that

$$Z(p) = \left(-\frac{2}{3} + \frac{2p_2}{3p_1}, \frac{2p_1}{3p_2} - \frac{2}{3} \right).$$

Setting $Z_1(p) = 0$ we obtain $p_1 = p_2$.
Setting $Z_2(p) = 0$ we obtain $p_1 = p_2$.

Equilibrium price is (λ, λ) for any $\lambda > 0$.

Note: if at p^*, we have $Z_1(p^*) = 0$ then $Z_2(p^*) = 0$. This follows from Walras' Law, which says that $p_1^* \, Z_1(p^*) + p_2^* \, Z_2(p^*) = 0$.

Exchange Economy

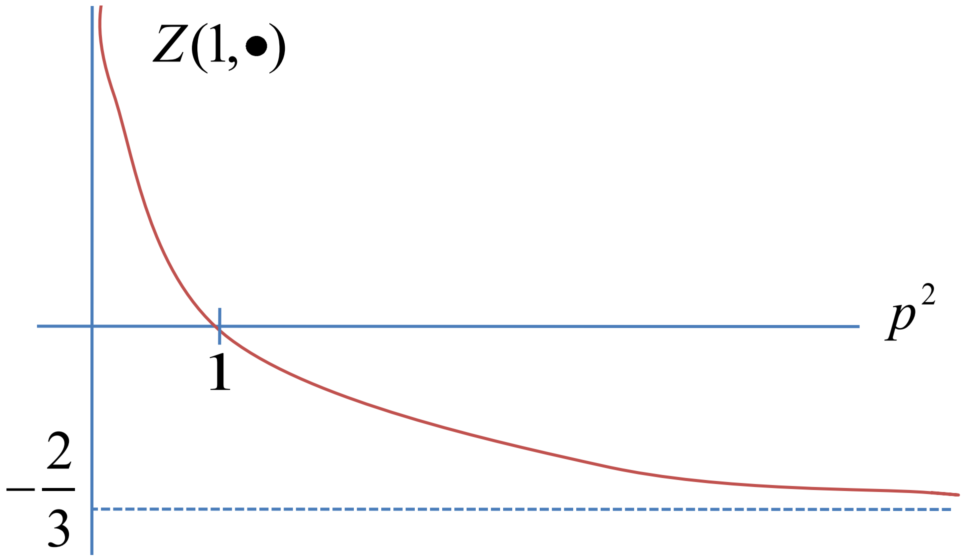
Exercise: show that

$$Z(p) = \left(-\frac{2}{3} + \frac{2p_2}{3p_1}, \frac{2p_1}{3p_2} - \frac{2}{3} \right).$$

Setting $Z_1(p) = 0$ we obtain $p_1 = p_2$.
Setting $Z_2(p) = 0$ we obtain $p_1 = p_2$.

Equilibrium price is (λ, λ) for any $\lambda > 0$.

Note: if at p^*, we have $Z_1(p^*) = 0$ then $Z_2(p^*) = 0$. This follows from Walras' Law, which says that $p_1^* Z_1(p^*) + p_2^* Z_2(p^*) = 0$.



Exchange Economy

Clear from picture that the existence of a solution to $Z(p) = 0$ requires the continuity of Z and also the right boundary condition ...

Continuity of Z is guaranteed if $\bar{x}^a$ is continuous for all a. This is in turn guaranteed by (P1) (the continuity of U^a for all agents a).

Turning to the boundary conditions...

Exchange Economy

Clear from picture that the existence of a solution to $Z(p) = 0$ requires the continuity of Z and also the right boundary condition ...

Continuity of Z is guaranteed if $\bar{x}^a$ is continuous for all a. This is in turn guaranteed by (P1) (the continuity of U^a for all agents a).

Turning to the boundary conditions...

Proposition: Suppose U^a obeys (P1), (P2), and (P3). Let the sequence $(p^n, w^n) \to (\bar{p}, \bar{w})$ such that $\bar{w} > 0$ and the set $I = \{i : \bar{p}_i = 0\}$ is nonempty. Then $\sum_{i \in I} \bar{x}_i^a(p^n, w^n) \to \infty$.

Exchange Economy

Clear from picture that the existence of a solution to $Z(p) = 0$ requires the continuity of Z and also the right boundary condition ...

Continuity of Z is guaranteed if $\bar{x}^a$ is continuous for all a. This is in turn guaranteed by (P1) (the continuity of U^a for all agents a).

Turning to the boundary conditions...

Proposition: Suppose U^a obeys (P1), (P2), and (P3). Let the sequence $(p^n, w^n) \to (\bar{p}, \bar{w})$ such that $\bar{w} > 0$ and the set $I = \{i : \bar{p}_i = 0\}$ is nonempty. Then $\sum_{i \in I} \bar{x}_i^a(p^n, w^n) \to \infty$.

Example: Cobb-Douglas demand for good j is $\alpha_j \frac{w^n}{p_j}$. Clearly tends to infinity if $p_j \to 0$ and $w^n \to \bar{w}$, with $\bar{w} > 0$.

Exchange Economy

Clear from picture that the existence of a solution to $Z(p) = 0$ requires the continuity of Z and also the right boundary condition ...

Continuity of Z is guaranteed if $\bar{x}^a$ is continuous for all a. This is in turn guaranteed by (P1) (the continuity of U^a for all agents a).

Turning to the boundary conditions...

Proposition: Suppose U^a obeys (P1), (P2), and (P3). Let the sequence $(p^n, w^n) \to (\bar{p}, \bar{w})$ such that $\bar{w} > 0$ and the set $I = \{i : \bar{p}_i = 0\}$ is nonempty. Then $\sum_{i \in I} \bar{x}_i^a(p^n, w^n) \to \infty$.

Example: Cobb-Douglas demand for good j is $\alpha_j \frac{w^n}{p_j}$. Clearly tends to infinity if $p_j \to 0$ and $w^n \to \bar{w}$, with $\bar{w} > 0$.

The requirement that $\bar{w} > 0$ is crucial. Compare $p^n = (1, \frac{1}{n})$ with $\omega = (1, 0)$ and $\omega = (0, 1)$; w^n tends to 1 in the first case and 0 in the second...

Exchange Economy

Corollary: In economy $\mathcal{E}$, suppose p^n tends to $\bar{p} \neq 0$, such that $I = \{i : \bar{p}_i = 0\}$ is nonempty. Then $\sum_{i \in I} Z_i(p^n) \to \infty$.

Exchange Economy

Corollary: In economy $\mathcal{E}$, suppose p^n tends to $\bar{p} \neq 0$, such that $I = \{i : \bar{p}_i = 0\}$ is nonempty. Then $\sum_{i \in I} Z_i(p^n) \to \infty$.

Proof: Suppose that for good k, $\bar{p}_k > 0$. Since $\bar{\omega} = \sum_{a \in A} \omega^a \gg 0$, there is $\tilde{a}$ with $\omega_k^{\tilde{a}} > 0$. (In other words, there is some agent $\tilde{a}$ who has a strictly positive endowment of good k.) Then $p^n \cdot \omega^{\tilde{a}}$ tends to $\bar{w}^{\tilde{a}} = \bar{p} \cdot \omega^{\tilde{a}} > 0$.

Exchange Economy

Corollary: In economy $\mathcal{E}$, suppose p^n tends to $\bar{p} \neq 0$, such that $I = \{i : \bar{p}_i = 0\}$ is nonempty. Then $\sum_{i \in I} Z_i(p^n) \to \infty$.

Proof: Suppose that for good k, $\bar{p}_k > 0$. Since $\bar{\omega} = \sum_{a \in A} \omega^a \gg 0$, there is $\tilde{a}$ with $\omega_k^{\tilde{a}} > 0$. (In other words, there is some agent $\tilde{a}$ who has a strictly positive endowment of good k.) Then $p^n \cdot \omega^{\tilde{a}}$ tends to $\bar{w}^{\tilde{a}} = \bar{p} \cdot \omega^{\tilde{a}} > 0$.

So $\sum_{i \in I} \hat{x}_i^{\tilde{a}}(p^n) = \sum_{i \in I} \bar{x}_i^{\tilde{a}}(p^n, p^n \cdot \omega^{\tilde{a}}) \to \infty$, which implies that

$$\begin{aligned} \sum_{i \in I} Z_i(p^n) &= \sum_{i \in I} X_i(p^n) - \sum_{i \in I} \bar{\omega}_i \\ &\geq \sum_{i \in I} \bar{x}_i^{\tilde{a}}(p^n, p^n \cdot \omega^{\tilde{a}}) - \sum_{i \in I} \bar{\omega}_i \to \infty. \end{aligned}$$

QED

Exchange Economy

Corollary: In economy $\mathcal{E}$, suppose p^n tends to $\bar{p} \neq 0$, such that $I = \{i : \bar{p}_i = 0\}$ is nonempty. Then $\sum_{i \in I} Z_i(p^n) \to \infty$.

Proof: Suppose that for good k, $\bar{p}_k > 0$. Since $\bar{\omega} = \sum_{a \in A} \omega^a \gg 0$, there is $\tilde{a}$ with $\omega_k^{\tilde{a}} > 0$. (In other words, there is some agent $\tilde{a}$ who has a strictly positive endowment of good k.) Then $p^n \cdot \omega^{\tilde{a}}$ tends to $\bar{w}^{\tilde{a}} = \bar{p} \cdot \omega^{\tilde{a}} > 0$.

So $\sum_{i \in I} \hat{x}_i^{\tilde{a}}(p^n) = \sum_{i \in I} \bar{x}_i^{\tilde{a}}(p^n, p^n \cdot \omega^{\tilde{a}}) \to \infty$, which implies that

$$\begin{aligned}
\sum_{i \in I} Z_i(p^n) &= \sum_{i \in I} X_i(p^n) - \sum_{i \in I} \bar{\omega}_i \\
&\geq \sum_{i \in I} \bar{x}_i^{\tilde{a}}(p^n, p^n \cdot \omega^{\tilde{a}}) - \sum_{i \in I} \bar{\omega}_i \to \infty.
\end{aligned}$$

QED

Exercise: In the example, let $p^n = (1, 1/n)$. Show that $x_2^A(p^n)$ tends to infinity and that $Z_2(p^n)$ tends to infinity. Show also that $x_2^B(p^n)$ does not tend to infinity; why?

Exchange Economy

Theorem: The excess demand function $Z : R^l_{++} \rightarrow R^l$ of the economy $\mathcal{E}$ has the following properties:

- it is zero-homogenous,

Exchange Economy

Theorem: The excess demand function $Z : R^l_{++} \to R^l$ of the economy $\mathcal{E}$ has the following properties:

- it is zero-homogenous,
- it obeys Walras' Law,

Exchange Economy

Theorem: The excess demand function $Z : R^l_{++} \to R^l$ of the economy $\mathcal{E}$ has the following properties:

- it is zero-homogenous,
- it obeys Walras' Law,
- it is continuous,

Exchange Economy

Theorem: The excess demand function $Z : R^l_{++} \rightarrow R^l$ of the economy $\mathcal{E}$ has the following properties:

- it is zero-homogenous,
- it obeys Walras' Law,
- it is continuous,
- it satisfies the boundary condition,

Exchange Economy

Theorem: The excess demand function $Z : R^l_{++} \rightarrow R^l$ of the economy $\mathcal{E}$ has the following properties:

- it is zero-homogenous,
- it obeys Walras' Law,
- it is continuous,
- it satisfies the boundary condition,
- it is bounded below.

Exchange Economy

Theorem: The excess demand function $Z : R^l_{++} \rightarrow R^l$ of the economy $\mathcal{E}$ has the following properties:

- it is zero-homogenous,
- it obeys Walras' Law,
- it is continuous,
- it satisfies the boundary condition,
- it is bounded below.

Note: Clear that Z is bounded below since

$$Z(p) = X(p) - \bar{\omega} > -\bar{\omega}.$$

Exchange Economy

Theorem: The excess demand function $Z : R^l_{++} \to R^l$ of the economy $\mathcal{E}$ has the following properties:

- it is zero-homogenous,
- it obeys Walras' Law,
- it is continuous,
- it satisfies the boundary condition,
- it is bounded below.

Note: Clear that Z is bounded below since

$$Z(p) = X(p) - \bar{\omega} > -\bar{\omega}.$$

Exchange Economy

Theorem (Arrow and Debreu; McKenzie): Suppose Z satisfies properties (1) to (5). Then there is p^* such that $Z(p^*) = 0$.

Proof uses Kakutani's Fixed Point Theorem.

Exchange Economy

When the economy has just two goods, existence can be proved using the intermediate value theorem.

Consider the function $Z_2(1, \cdot) : R_{++} \to R$.

Exchange Economy

When the economy has just two goods, existence can be proved using the intermediate value theorem.

Consider the function $Z_2(1, \cdot) : R_{++} \to R$.

First note that as $p \to 0$, we have $Z_2(1, p) \to \infty$, so there is p' such that $Z_2(1, p') > 0$.

Exchange Economy

When the economy has just two goods, existence can be proved using the intermediate value theorem.

Consider the function $Z_2(1, \cdot) : R_{++} \to R$.

First note that as $p \to 0$, we have $Z_2(1, p) \to \infty$, so there is p' such that $Z_2(1, p') > 0$.

Now consider $p \to \infty$.

Then $Z_1(1, p) = Z_1(\frac{1}{p}, 1) \to \infty$. In particular, there is p'' such that $Z_1(1, p'') > 0$. This implies that $Z_2(1, p'') < 0$ (by Walras' Law).

Exchange Economy

When the economy has just two goods, existence can be proved using the intermediate value theorem.

Consider the function $Z_2(1, \cdot) : R_{++} \to R$.

First note that as $p \to 0$, we have $Z_2(1, p) \to \infty$, so there is p' such that $Z_2(1, p') > 0$.

Now consider $p \to \infty$.

Then $Z_1(1, p) = Z_1(\frac{1}{p}, 1) \to \infty$. In particular, there is p'' such that $Z_1(1, p'') > 0$. This implies that $Z_2(1, p'') < 0$ (by Walras' Law).

So there is p' and p'' such that $Z_2(1, p') > 0$ and $Z_2(1, p'') < 0$.By IVT, there is p^* such that $Z_2(1, p^*) = 0$. **QED**