# *Lectures on General Equilibrium Theory*

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# *The first and second welfare theorems*

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# Pareto optimality

An allocation attributes a bundle of goods to each agent in the economy $\mathcal{E}$. Formally, an allocation is a map $\phi : A \to R^l_+$, where $\phi(a)$ is the bundle given to agent $a$.

We may also write an allocation by $\{y^a\}_{a\in A}$, where $y^a \in R^l_+$ is the bundle given to agent $a$.

An allocation $\{y^a\}_{a\in A}$ is said to be feasible if $\sum_{a\in A} y^a = \bar{\omega}$.

Suppose $p^* \in R^l_{++}$ is an equilibrium price in the economy $\mathcal{E}$. Then $\sum_{a\in A} \hat{x}^a(p^*) = \bar{\omega}$. So the Walrasian allocation $\{\hat{x}^a(p^*)\}_{a\in A}$ is a feasible allocation.

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An allocation $\{z^a\}_{a\in A}$ is a Pareto improvement of another allocation $\{y^a\}_{a\in A}$ if $U^a(z^a) \geq U^a(y^a)$ for all $a \in A$ and the inequality is strict for at least one agent.

A feasible allocation $\{y^a\}_{a\in A}$ is Pareto optimal if it cannot be Pareto-improved by another feasible allocation.

# First welfare theorem

Theorem: Suppose $U^a$ is increasing for all $a$. Then every Walrasian allocation is Pareto optimal.

Recall that a utility function $U^a : R^l_+ \to R$ is monotone if, whenever $y \gg x$, then $U^a(y) > U^a(x)$.

Note that since the utility function $U^a$ depends only on what $a$ consumes, externalities are excluded.

If the consumption of another agent $b$ affects $a$'s utility, then $a$'s utility function will depend on $x^a$ and $x^b$ (agent $a$ and $b$'s consumption respectively); for example,

$$U^a(x^a, x^b) = U^a(x^a) + V(x^b).$$

# First Welfare Theorem

Theorem: Suppose $U^a$ is monotone for all $a$. Then every Walrasian allocation is Pareto optimal.

Proof: We prove by contradiction. Suppose that $\{z^a\}_{a \in A}$ is a Pareto improvement of $\{\hat{x}^a(p^*)\}_{a \in A}$, so $U^a(z^a) \geq U^a(\hat{x}^a(p^*))$ for all $a \in A$ with a strict inequality for $\tilde{a}$.

By definition, $\hat{x}^{\tilde{a}}(p^*)$ maximizes agent $\tilde{a}$'s utility in $B(p^*, p^* \cdot \omega^{\tilde{a}})$. So the bundle $z^{\tilde{a}}$ cannot be affordable to agent $\tilde{a}$, i.e.,

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Furthermore, I claim that $p^* \cdot z^a \geq p^* \cdot \omega^a$ for all agents $a$.

Suppose $p^* \cdot z^a < p^* \cdot \omega^a$ for some agent $a$. Then there exists $\epsilon > 0$ such that

$$p^* \cdot (z^a + (\epsilon, \epsilon, ..., \epsilon)) < p^* \cdot \omega^a.$$

Since $U^a$ is increasing,

$$U^a(z^a + (\epsilon, \epsilon, ..., \epsilon)) > U^a(z^a) \geq U^a(\hat{x}^a(p^*))$$

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We conclude that $p^* \cdot \left[\sum_{a\in A} z^a\right] > p^* \cdot \left[\sum_{a\in A} \omega^a\right]$, which cannot be true since $\sum_{a\in A} z^a = \sum_{a\in A} \omega^a$.

# Second Welfare Theorem

An allocation $\{y^a\}_{a \in A}$ is a Walrasian allocation with transfers if it is feasible and there is $p^* \gg 0$ and $\{t^a\}_{a \in A}$ with $\sum_{a \in A} t^a = 0$ (where $t^a \in R$) such that

$$\bar{x}^a(p^*, p^* \cdot \omega^a + t^a) = y^a.$$

We refer to $t^a$ as the transfer received by agent $a$.

The transfers are balanced since they add up to zero – some agents receive a transfer, others are taxed.

There is a price vector $p^*$ such that agent $a$'s demand at price $p^*$ and wealth $p^* \cdot \omega^a + t^a$ is precisely $y^a$ (for every agent $a$).

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Theorem: Suppose that $U^a$ is strictly increasing, strictly quasiconcave, and continuous for all $a$. Then every Pareto optimal allocation is a Walrasian allocation with transfers.

The theorem says that given a Pareto optimal allocation $\{y^a\}_{a \in A}$, there is a price $p^*$ and an income transfer to agent $a$, $t_a$, such that

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Note that so long as the utility function of $a$ is quasiconcave, there is a price $p^a$ and an income $m^a$ such that $\bar{x}^a(p^a, m^a) = y^a$.

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(The price $p^a$ is called a supporting price.) What is remarkable here is that $p^a$ is the *same across agents*.

Loosely speaking, at any Pareto optimal allocation, the indifference surface of agent $a$ at $y^a$ has the same slope as that of agent $\tilde{a}$ at $y^{\tilde{a}}$. Usual Edgeworth box depiction...

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Proof: Let $\{y^a\}_{a\in A}$ be a Pareto optimal allocation. Consider the economy where agent $a$ has endowment $y^a$. Given the properties of $U^a$, the equilibrium existence theorem is applicable. So there is $p^* \gg 0$ which is the equilibrium price of this economy. Therefore,

$$\sum_{a\in A} \bar{x}^a(p^*, p^* \cdot y^a) = \sum_{a\in A} y^a = \bar{\omega}.$$

Note that $u^a(\bar{x}^a(p^*, p^* \cdot y^a)) \geq u^a(y^a)$ for all $a$.

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$$\sum_{a\in A} t^a = p^* \cdot \left( \sum_{a\in A} y^a - \sum_{a\in A} \omega^a \right) = 0$$