Lectures on General Equilibrium Theory Michaelmas 2008

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Weak Axiom of Revealed Preference

Agent a's demand function $\bar{x}^a$ obeys the weak axiom of revealed preference if at any (p, w) and (p', w') with $\bar{x}^a(p, w) \neq \bar{x}(p', w')$, the following holds:

$$p' \cdot \bar{x}^a(p, w) \leq w' \Longrightarrow p \cdot \bar{x}^a(p', w') > w.$$

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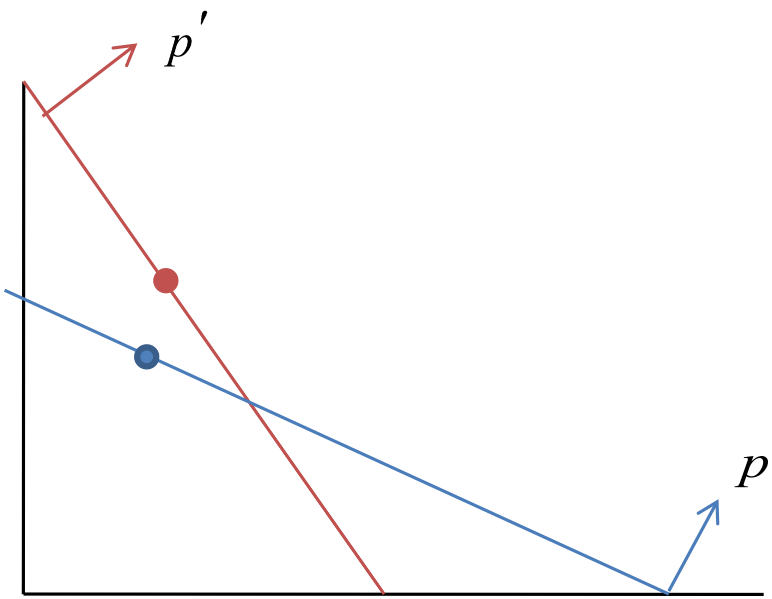

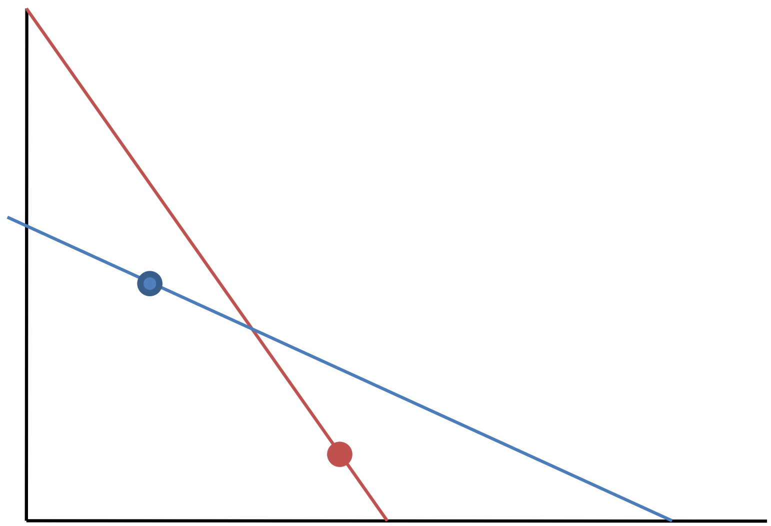

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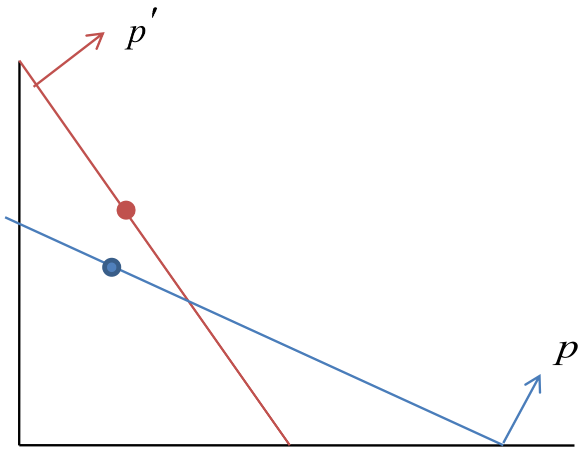

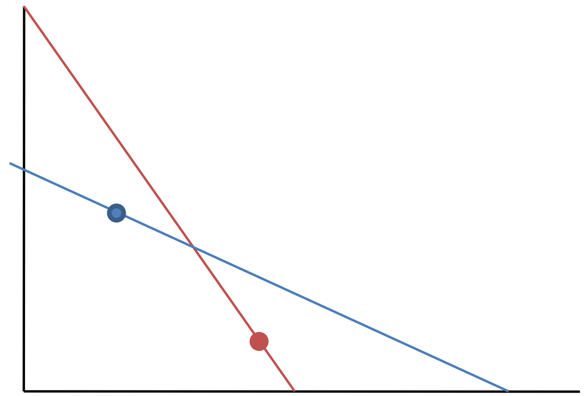

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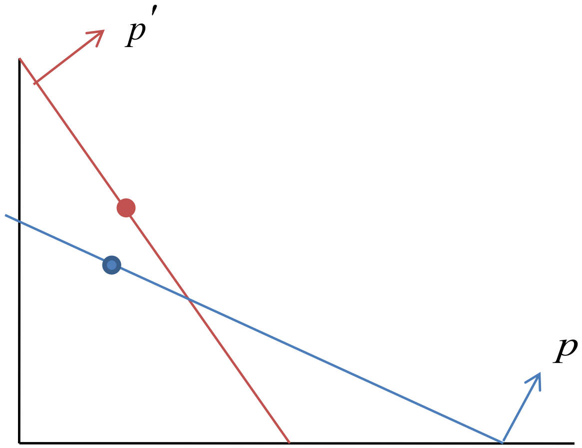

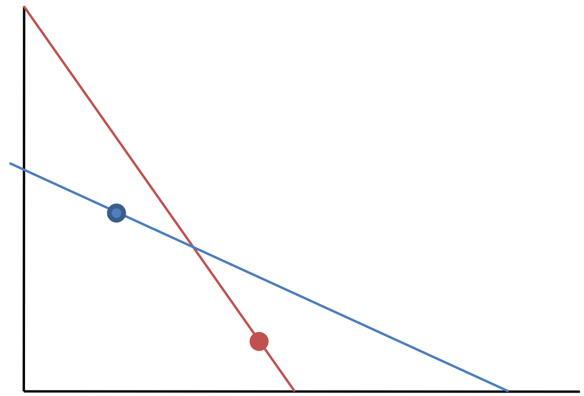

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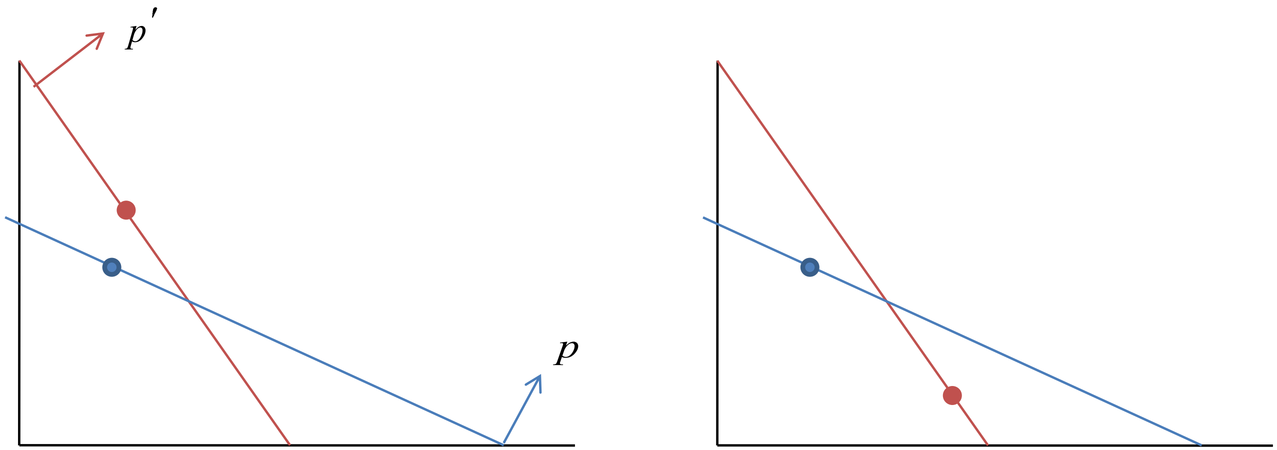


Proposition: Agent a's demand function obeys the weak axiom of revealed preference if a is utility-maximizing.

Proof: If $p' \cdot \bar{x}^a(p,w) \leq w'$ then $x^a(p,w)$ is in $B(p',w')$. But $\bar{x}^a(p,w)$ is not the demand at (p',w') so

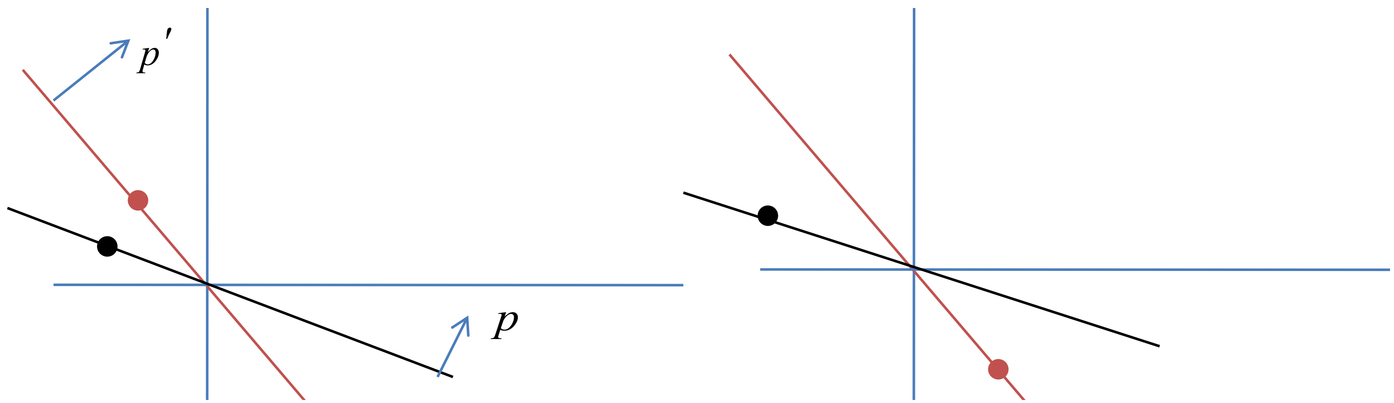
$$U(\bar{x}^a(p',w')) > U(\bar{x}^a(p,w)).$$

Thus $\bar{x}^a(p',w') \notin B(p,w)$; otherwise it would be chosen over $\bar{x}^a(p,w)$. In other words, $p \cdot \bar{x}^a(p',w') > w$. **QED**

Weak Axiom of Revealed Preference

Corollary: Agent a's excess demand function z^a obeys the weak axiom of revealed preference: at prices p and p', if $z^a(p) \neq z^a(p')$, then

$$p' \cdot z^a(p) \leq 0 \Longrightarrow p \cdot z^a(p') > 0.$$



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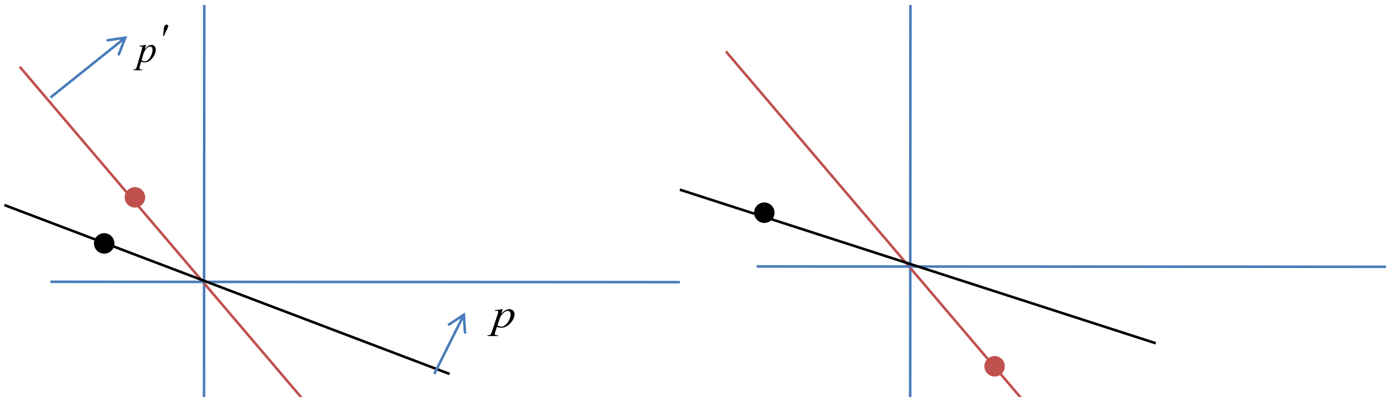

Proof: Since $z^a(p) = \bar{x}^a(p, p \cdot \omega^a)$, we may re-write $p' \cdot z^a(p) \leq 0$ as $p' \cdot \bar{x}^a(p, p \cdot \omega^a) \leq p' \cdot \omega^a$. Set $p \cdot \omega^a = w$ and $p' \cdot \omega^a = w'$. By previous result,

$$p \cdot \bar{x}^a(p', w') > w = p \cdot \omega^a.$$

Re-write this as $p \cdot z^a(p') > 0$. **QED**

Structure of excess demand function Z

What is the structure of $Z(p) = \sum_{a \in A} z^a(p)$?

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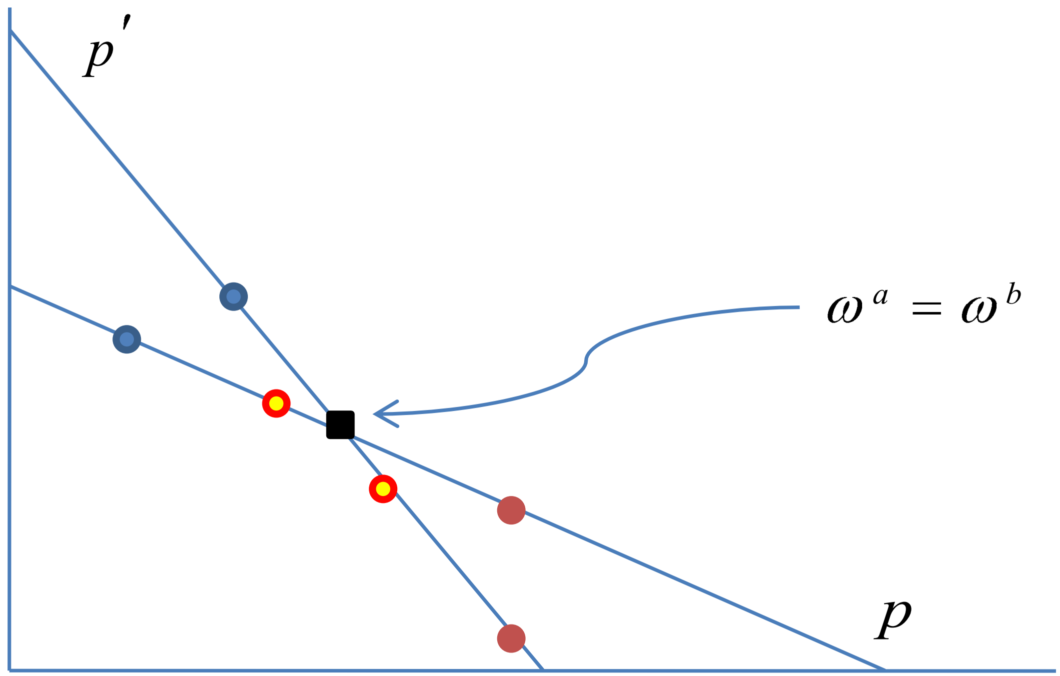

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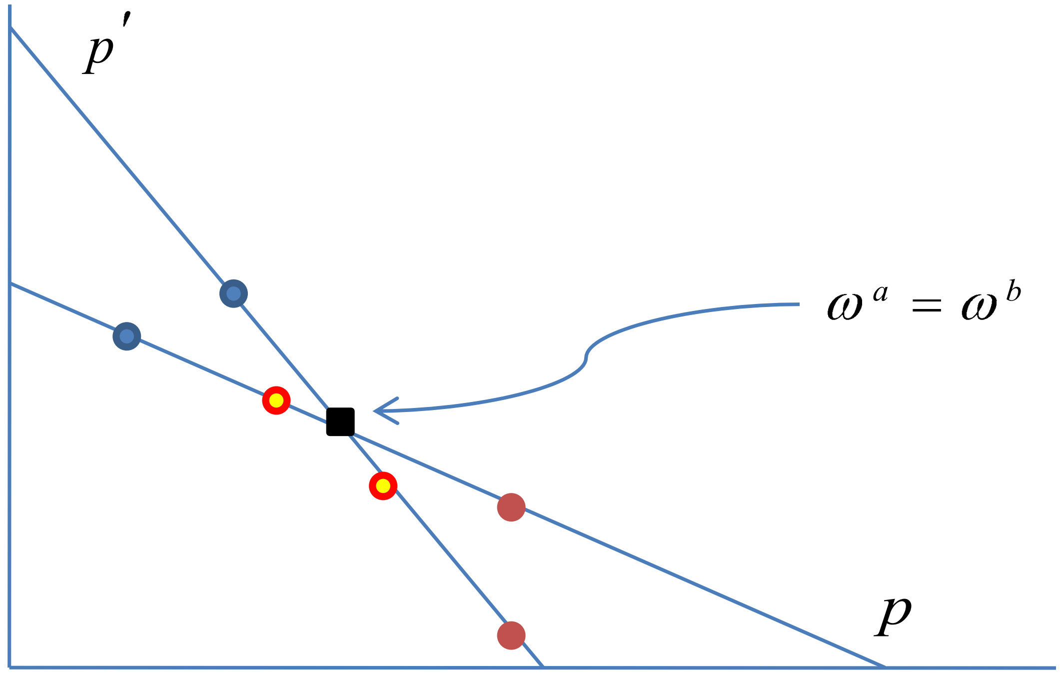

The aggregate excess demand function Z need not obey the weak axiom.

Structure of excess demand function Z

Recall (from earlier theorem) that the excess demand function $Z : R^l_{++} \to R^l$ of the economy has the following properties: it is zero-homogenous, it obeys Walras' Law, it is continuous, it satisfies the boundary condition, and it is bounded below.

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Can anything more be said about Z?

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Can anything more be said about Z?

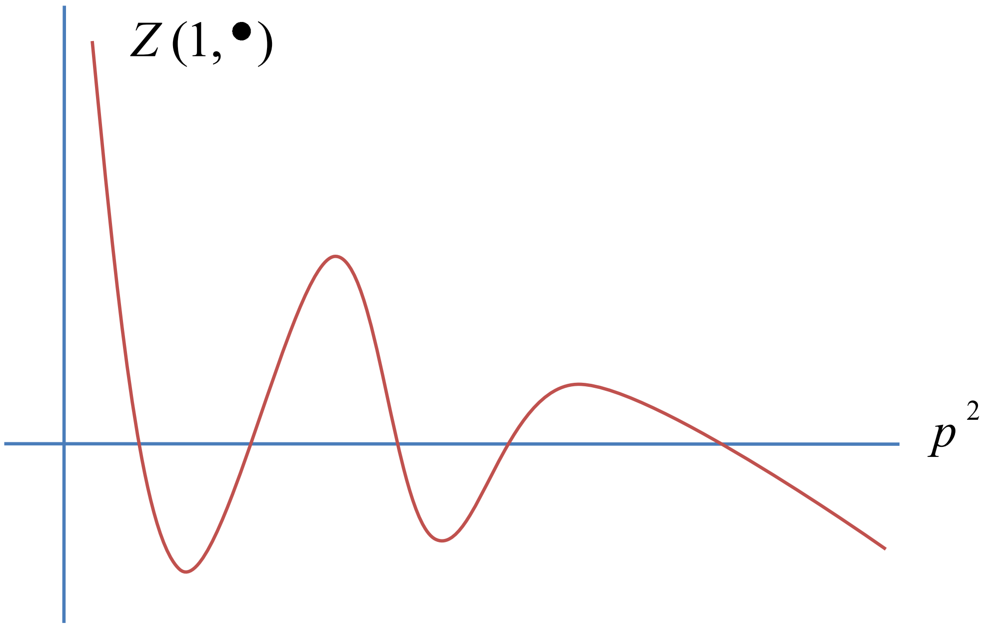
Indeterminacy Theorem (Sonnenschein-Mantel-Debreu): Let P be a compact set in R^l_{++} and let $S : P \to R$ be a function with the following properties: it is zero-homogenous, it obeys Walras' Law, and it is continuous.

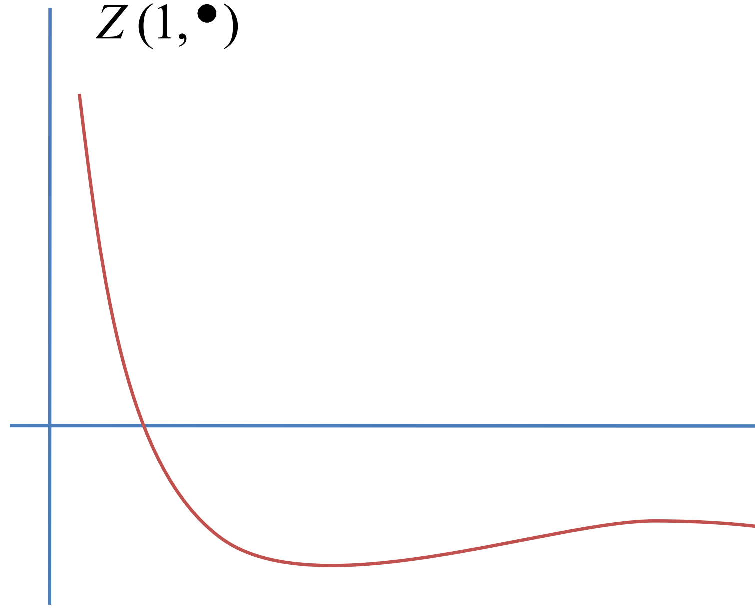
There is an exchange economy of agents with utility functions obeying (P1), (P2), and (P3) such that its excess demand function $Z : R^l_{++} \to R^l$ satisfies

$$Z(p) = S(p) \text{ for all } p \in P.$$

Structure of excess demand function Z

Corollary: Multiple equilibria and unstable equilibria are possible.





Demand Aggregation

Maybe it isn't so bad after all... there *is* structure to Z if stronger restrictions are imposed on utility functions and endowments.

Two types of aggregate structure on Z widely studied:

gross substitutability and the weak axiom.

Demand Aggregation

Theorem: Suppose that all agents in $\mathcal{E}$ have the same utility function and thus the same demand function, so

$$\bar{x}^a(p, w) = \bar{x}(p, w) \text{ for all } a \in A.$$

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(Demand is linear if preference is homothetic. Such a preference is representable by a 1-homogeneous utility function, i.e., $U(tx) = tU(x)$ for any $t > 0$ and x in R^l_+.)

Demand Aggregation

Proof: Denote the economy's aggregate endowment $\sum_{a \in A} \omega^a$ by $\bar{\omega}$.
Then $X(p) = \sum_{a \in A} \bar{x}(p, p \cdot \omega^a)$

$$= \sum_{a \in A} \bar{x}(p, 1)(p \cdot \omega^a)$$

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$Z(p)=\bar{x}(p,p\cdot\bar{\omega})-\bar{\omega}$ obeys the weak axiom. **QED**

Weak Axiom at Equilibrium

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Let p^* be an equilibrium of $\mathcal{E}$. We say that its excess demand function Z obeys the weak axiom at equilibrium if $(p - p^*) \cdot Z(p) < 0$ for all p not collinear with p^*.

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Let p^* be an equilibrium of $\mathcal{E}$. We say that its excess demand function Z obeys the weak axiom at equilibrium if $(p - p^*) \cdot Z(p) < 0$ for all p not collinear with p^*.

By proposition, Z obeys WAE if it obeys WA and has a unique equilibrium price.

Weak Axiom at Equilibrium

In fact, if Z obeys the weak axiom then the set of equilibrium prices is convex, i.e., if p^* and p^{**} are equilibrium prices, so is $tp^* + (1-t)p^{**}$ for t in $[0, 1]$.

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So when Z obeys the WA, generically, the price equilibrium is *unique* and Z obeys WAE.

Interpretation of WAE: if $p = (p_1, p_2^*, p_3^*, ..., p_l^*)$, then

$$(p - p^*) \cdot Z(p) = (p_1 - p_1^*) Z_1(p) < 0.$$

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When price of 1 is higher than its equilibrium price, there is excess supply of 1; etc.

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Not generally. Depends on Z.

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Lemma: Let $p(t)$ be the solution to Walras tatonnement at the initial condition $p(0) = \bar{p}$. Then

$$\sum_{i=1}^{l} p_i^2(t) = \sum_{i=1}^{l} \bar{p}_i^2 \text{ for all } t.$$

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$$\frac{d}{dt}\left(\sum_{i=1}^{l} p_i^2(t)\right) = 2\left(\sum_{i=1}^{l} p_i(t)\frac{dp_i}{dt}(t)\right) = 2\left(\sum_{i=1}^{l} p_i(t) Z_i(p(t))\right).$$

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Thus, the solution $p(t)$ lies on the surface of a higher dimensional sphere with radius $\sqrt{\sum_{i=1}^{l} \bar{p}_i^{\ 2}}$.

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Consider the Lyapunov function $L(p) = \sum_{i=1}^{l} (p_i - p_i^*)^2$. Then

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The latter equals $2(p - p^*) \cdot Z(p)$, so $\frac{dL}{dt} < 0$. **QED**