Lectures on General Equilibrium Theory

⋆ ⋆ ⋆

Demand aggregation and the structure of equilibrium

John Quah

john.quah@economics.ox.ac.uk

Weak Axiom of Revealed Preference

A function $F : R^l_{++} \to R^l$ satisfies the weak axiom of revealed preference if at any p and p' with $F(p) \neq F(p')$, the following holds:

$$p' \cdot F(p) \leq p' \cdot F(p') \Longrightarrow p \cdot F(p') > p \cdot F(p).$$

In the case where $F(p) = \bar{x}^a(p, w)$ (the demand of agent a), WARP says that at any p and p' with $\bar{x}^a(p, w) \neq \bar{x}(p', w)$, the following holds:

$$p' \cdot \bar{x}^a(p, w) \leq p' \cdot \bar{x}^a(p', w) = w \Longrightarrow p \cdot \bar{x}^a(p', w) > p \cdot \bar{x}^a(p, w) = w.$$

Equivalently, $\bar{x}^a(p, w) \in B(p', w) \Longrightarrow \bar{x}^a(p', w') \notin B(p, w)$.

Weak Axiom of Revealed Preference

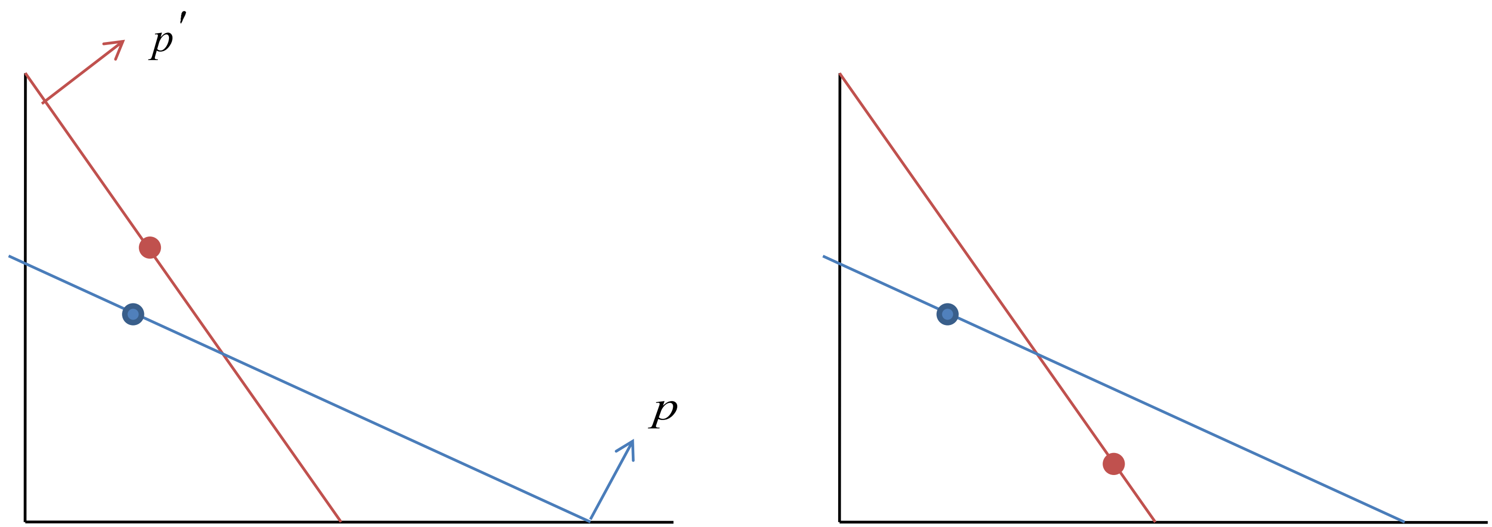
A function $F : R^l_{++} \to R^l$ satisfies the weak axiom of revealed preference if at any p and p' with $F(p) \neq F(p')$, the following holds:

$$p' \cdot F(p) \leq p' \cdot F(p') \Longrightarrow p \cdot F(p') > p \cdot F(p).$$

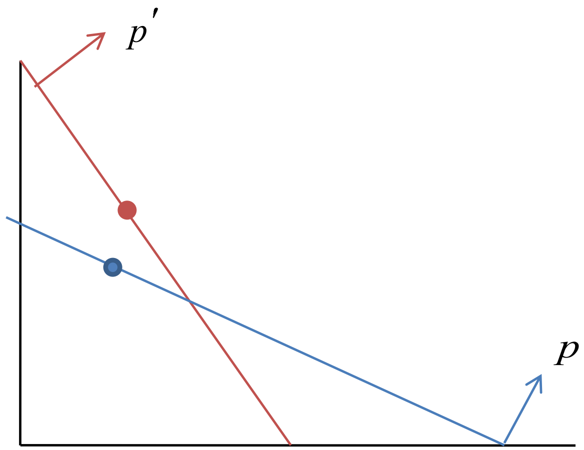
In the case where $F(p) = \bar{x}^a(p, w)$ (the demand of agent a), WARP says that at any p and p' with $\bar{x}^a(p, w) \neq \bar{x}(p', w)$, the following holds:

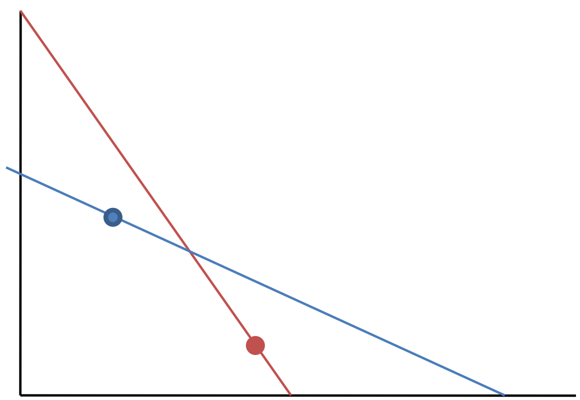
$$p' \cdot \bar{x}^a(p, w) \leq p' \cdot \bar{x}^a(p', w) = w \Longrightarrow p \cdot \bar{x}^a(p', w) > p \cdot \bar{x}^a(p, w) = w.$$

Equivalently, $\bar{x}^a(p, w) \in B(p', w) \Longrightarrow \bar{x}^a(p', w') \notin B(p, w)$.

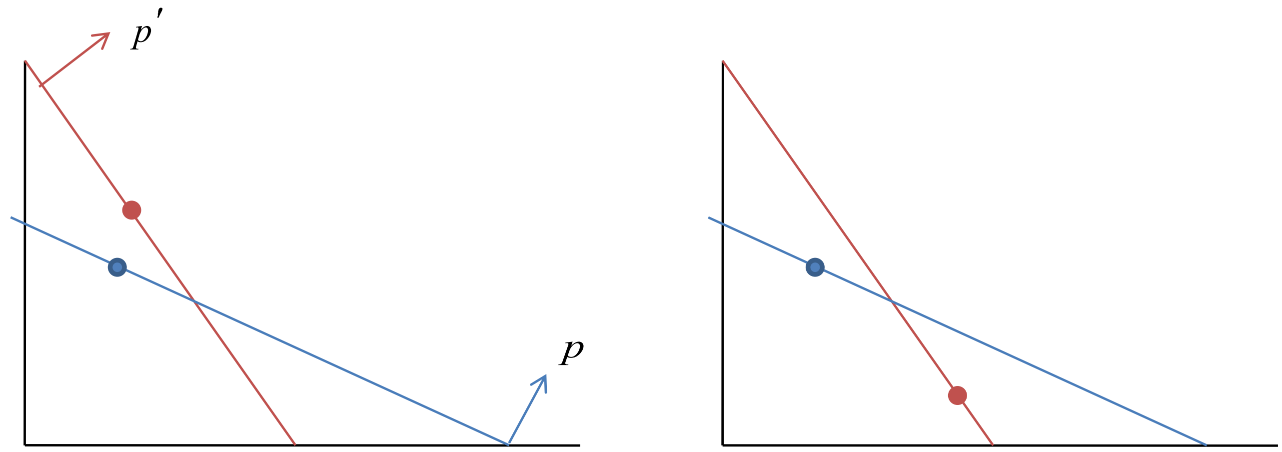


Weak Axiom of Revealed Preference



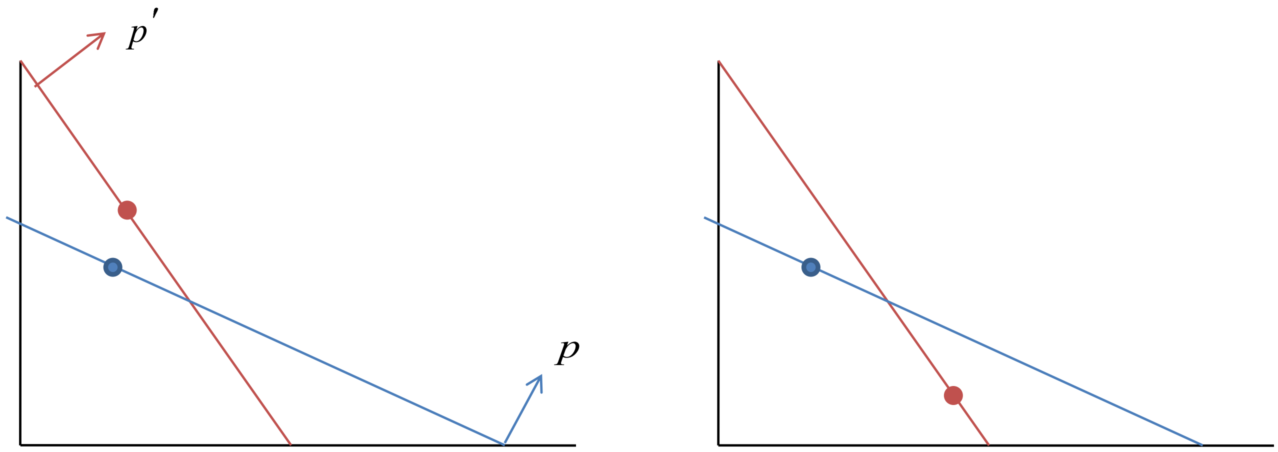


Weak Axiom of Revealed Preference



Proposition: Agent a's demand function obeys the weak axiom of revealed preference if a is utility-maximizing.

Weak Axiom of Revealed Preference



Proposition: Agent a's demand function obeys the weak axiom of revealed preference if a is utility-maximizing.

Proof: If $p' \cdot \bar{x}^a(p,w) \leq w$ then $x^a(p,w)$ is in $B(p',w)$. But $\bar{x}^a(p,w)$ is not the demand at (p',w) so

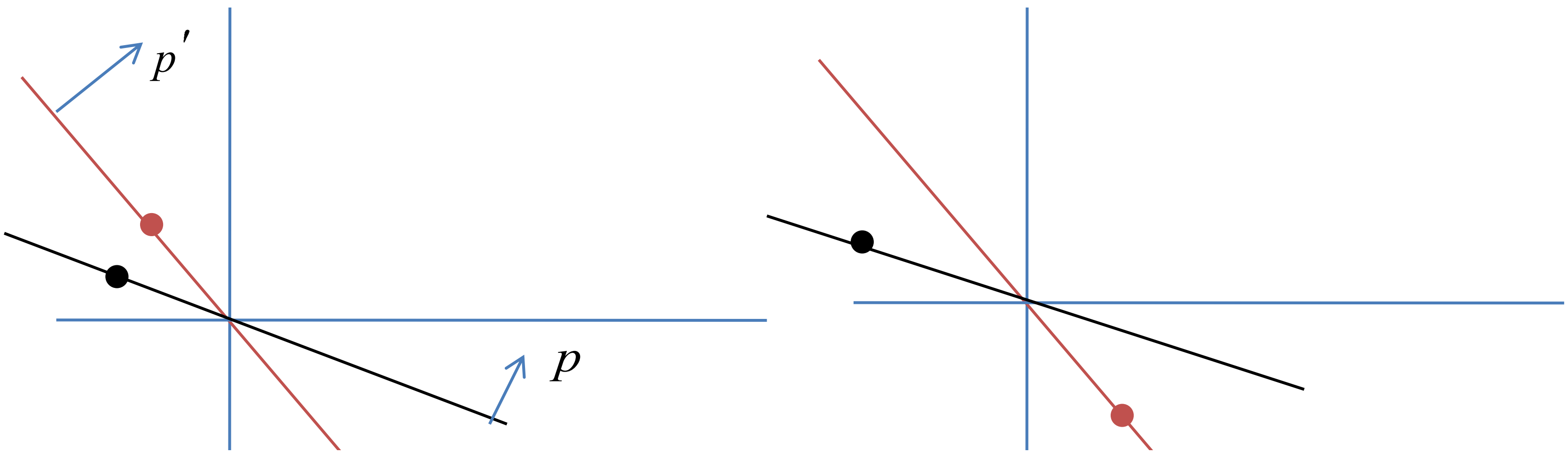
$$U(\bar{x}^a(p',w)) > U(\bar{x}^a(p,w)).$$

Thus $\bar{x}^a(p',w) \notin B(p,w)$; otherwise it would be chosen over $\bar{x}^a(p,w)$. In other words, $p \cdot \bar{x}^a(p',w) > w$. **QED**

Weak Axiom of Revealed Preference

Corollary: Agent a's excess demand function z^a obeys the weak axiom of revealed preference: at prices p and p', if $z^a(p) \neq z^a(p')$, then

$$p' \cdot z^a(p) \leq 0 \Longrightarrow p \cdot z^a(p') > 0.$$



Weak Axiom of Revealed Preference

Corollary: Agent a's excess demand function z^a obeys the weak axiom of revealed preference: at prices p and p', if $z^a(p) \neq z^a(p')$, then

$$p' \cdot z^a(p) \leq 0 \Longrightarrow p \cdot z^a(p') > 0.$$

Weak Axiom of Revealed Preference

Corollary: Agent a's excess demand function z^a obeys the weak axiom of revealed preference: at prices p and p', if $z^a(p) \neq z^a(p')$, then

$$p' \cdot z^a(p) \leq 0 \Longrightarrow p \cdot z^a(p') > 0.$$

Proof: Choose $\lambda > 0$ such that $p'' = \lambda p'$ satisfies $p \cdot \omega^a = p'' \cdot \omega^a = w$.

Since $z^a(p) = \bar{x}^a(p, p \cdot \omega^a) - \omega^a$, we may re-write $p' \cdot z^a(p) \leq 0$ as $p' \cdot \bar{x}^a(p, p \cdot \omega^a) \leq p' \cdot \omega^a$; equivalently,

$$p'' \cdot \bar{x}^a(p, w) \leq p'' \cdot \omega^a = w.$$

Weak Axiom of Revealed Preference

Corollary: Agent a's excess demand function z^a obeys the weak axiom of revealed preference: at prices p and p', if $z^a(p) \neq z^a(p')$, then

$$p' \cdot z^a(p) \leq 0 \Longrightarrow p \cdot z^a(p') > 0.$$

Proof: Choose $\lambda > 0$ such that $p'' = \lambda p'$ satisfies $p \cdot \omega^a = p'' \cdot \omega^a = w$.

Since $z^a(p) = \bar{x}^a(p, p \cdot \omega^a) - \omega^a$, we may re-write $p' \cdot z^a(p) \leq 0$ as $p' \cdot \bar{x}^a(p, p \cdot \omega^a) \leq p' \cdot \omega^a$; equivalently,

$$p'' \cdot \bar{x}^a(p, w) \leq p'' \cdot \omega^a = w.$$

Since $\bar{x}^a$ obeys the weak axiom,

$$p \cdot \bar{x}^a(p'', w) > w = p \cdot \omega^a.$$

Re-write this as $p \cdot z^a(p'') > 0$. Since $z^a(p'') = z^a(p')$, we obtain $p \cdot z^a(p') > 0$. **QED**

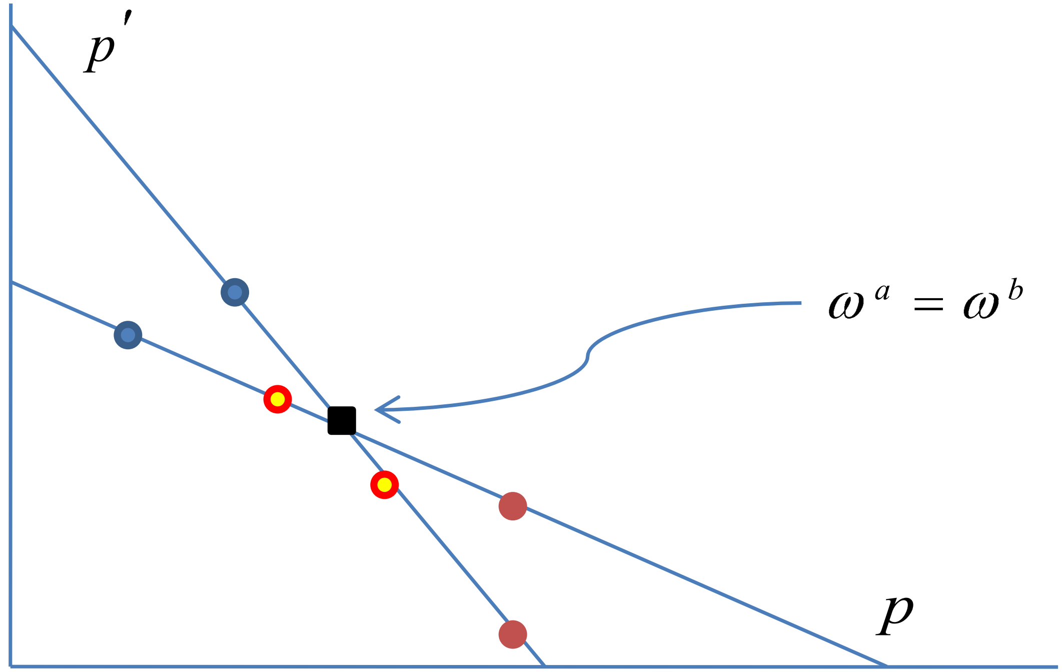
Structure of excess demand function Z

What is the structure of $Z(p) = \sum_{a \in A} z^a(p)$?

Structure of excess demand function Z

What is the structure of $Z(p) = \sum_{a \in A} z^a(p)$?

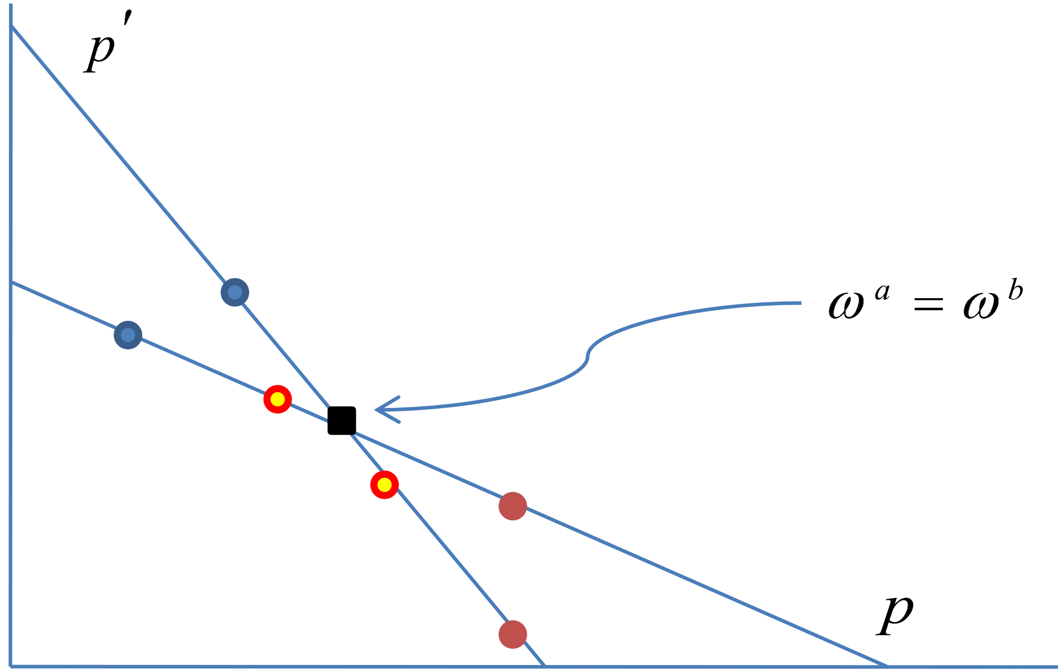
Hicks's example



Structure of excess demand function Z

What is the structure of $Z(p) = \sum_{a \in A} z^a(p)$?

Hicks's example



The aggregate excess demand function Z need not obey the weak axiom.

Structure of excess demand function Z

Recall (from earlier theorem) that the excess demand function $Z : R^l_{++} \to R^l$ of the economy has the following properties: it is zero-homogenous, it obeys Walras' Law, it is continuous, it satisfies the boundary condition, and it is bounded below.

Structure of excess demand function Z

Recall (from earlier theorem) that the excess demand function $Z : R^l_{++} \to R^l$ of the economy has the following properties: it is zero-homogenous, it obeys Walras' Law, it is continuous, it satisfies the boundary condition, and it is bounded below.

Can anything more be said about Z?

Structure of excess demand function Z

Recall (from earlier theorem) that the excess demand function $Z : R^l_{++} \to R^l$ of the economy has the following properties: it is zero-homogenous, it obeys Walras' Law, it is continuous, it satisfies the boundary condition, and it is bounded below.

Can anything more be said about Z?

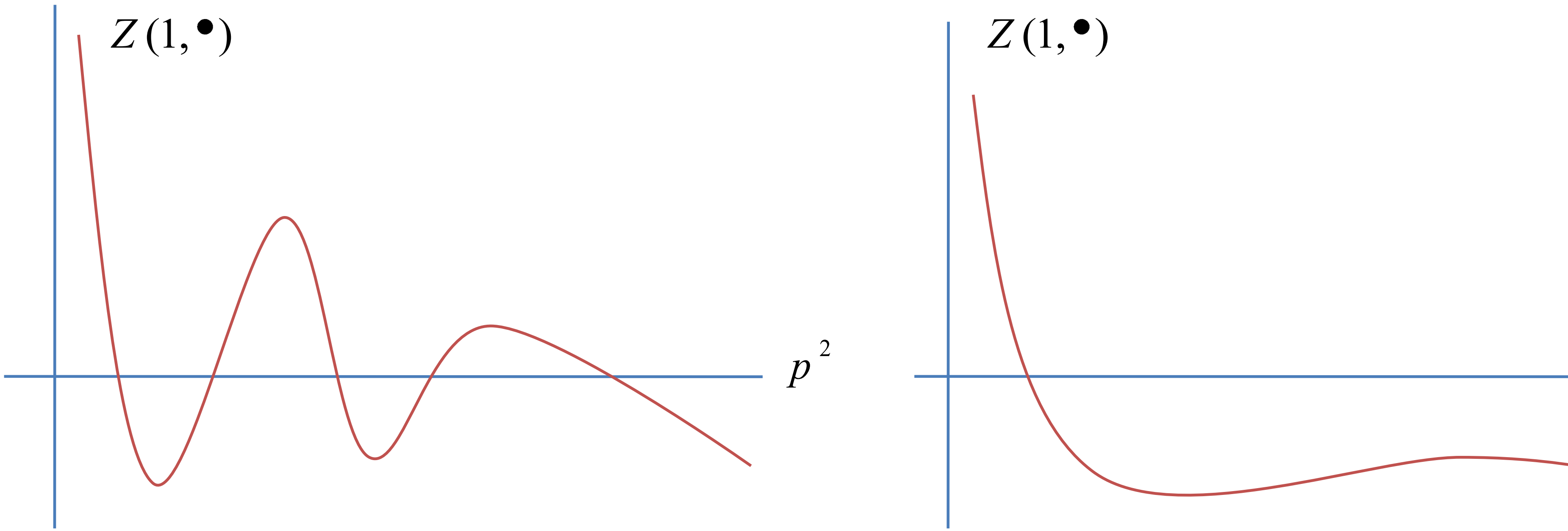
Indeterminacy Theorem (Sonnenschein-Mantel-Debreu): Let P be a compact set in R^l_{++} and let $S : P \to R^l$ be a function with the following properties: it is zero-homogenous, it obeys Walras' Law, and it is continuous.

There is an exchange economy of agents with utility functions obeying (P1), (P2), and (P3) such that its excess demand function $Z : R^l_{++} \to R^l$ satisfies

$$Z(p) = S(p) \text{ for all } p \in P.$$

Structure of excess demand function Z

Corollary: Multiple equilibria and unstable equilibria are possible.



Demand Aggregation

Maybe it isn't so bad after all... there *is* structure to Z if stronger restrictions are imposed on utility functions and endowments.

Two types of aggregate structure on Z widely studied:

gross substitutability and the weak axiom.

We shall examine simple conditions for the latter property.

Homothetic Preferences

A preference $\succeq$ is homothetic if $x \succeq x'$ implies that $\lambda x \succeq \lambda x'$ where $\lambda > 0$.

A demand function $\bar{x}$ is linear in income if $\bar{x}(p, \lambda w) = \lambda \bar{x}(p, w)$ for any $\lambda > 0$.

Proposition: A homothetic preference has a demand function that is linear in income.

Homothetic Preferences

A preference $\succeq$ is homothetic if $x \succeq x'$ implies that $\lambda x \succeq \lambda x'$ where $\lambda > 0$.

A demand function $\bar{x}$ is linear in income if $\bar{x}(p, \lambda w) = \lambda \bar{x}(p, w)$ for any $\lambda > 0$.

Proposition: A homothetic preference has a demand function that is linear in income.

Proof: Let $y = \bar{x}(p, w)$. If $x' \in B(p, \lambda w)$ then $x'/\lambda \in B(p, w)$.
So $y \succeq x'/\lambda$.

By the homotheticity of $\succeq$, we obtain $\lambda y \succeq x'$.

Therefore, $\lambda y = \bar{x}(p, \lambda w)$. **QED**

Demand Aggregation: example 1

Theorem: Suppose that all agents in $\mathcal{E}$ have the same utility function and thus the same demand function $\bar{x}$, with $\bar{x}(p, w) = \bar{x}(p, 1)w$, i.e., demand is linear in income. Then Z obeys the weak axiom.

Demand Aggregation: example 1

Theorem: Suppose that all agents in $\mathcal{E}$ have the same utility function and thus the same demand function $\bar{x}$, with $\bar{x}(p, w) = \bar{x}(p, 1)w$, i.e., demand is linear in income. Then Z obeys the weak axiom.

Proof: Denote the economy's aggregate endowment $\sum_{a \in A} \omega^a$ by $\bar{\omega}$.
Then $X(p) = \sum_{a \in A} \bar{x}(p, p \cdot \omega^a)$

$$= \sum_{a \in A} \bar{x}(p, 1)(p \cdot \omega^a)$$

$$= \bar{x}(p, 1) \left[\sum_{a \in A} (p \cdot \omega^a)\right]$$

Demand Aggregation: example 1

Theorem: Suppose that all agents in $\mathcal{E}$ have the same utility function and thus the same demand function $\bar{x}$, with $\bar{x}(p,w) = \bar{x}(p,1)w$, i.e., demand is linear in income. Then Z obeys the weak axiom.

Proof: Denote the economy's aggregate endowment $\sum_{a\in A}\omega^a$ by $\bar{\omega}$. Then $X(p) = \sum_{a\in A}\bar{x}(p, p\cdot\omega^a)$

$$= \sum_{a\in A}\bar{x}(p,1)(p\cdot\omega^a)$$

$$= \bar{x}(p,1)\left[\sum_{a\in A}(p\cdot\omega^a)\right]$$

$$= \bar{x}(p,1)\left[p\cdot\left(\sum_{a\in A}\omega^a\right)\right]$$

Demand Aggregation: example 1

Theorem: Suppose that all agents in $\mathcal{E}$ have the same utility function and thus the same demand function $\bar{x}$, with $\bar{x}(p, w) = \bar{x}(p, 1)w$, i.e., demand is linear in income. Then Z obeys the weak axiom.

Proof: Denote the economy's aggregate endowment $\sum_{a \in A} \omega^a$ by $\bar{\omega}$. Then $X(p) = \sum_{a \in A} \bar{x}(p, p \cdot \omega^a)$

$$= \sum_{a \in A} \bar{x}(p, 1)(p \cdot \omega^a)$$

$$= \bar{x}(p, 1) \left[\sum_{a \in A}(p \cdot \omega^a)\right]$$

$$= \bar{x}(p, 1) \left[p \cdot (\sum_{a \in A} \omega^a)\right]$$

$$= \bar{x}(p, p \cdot (\sum_{a \in A} \omega^a))$$

Demand Aggregation: example 1

Theorem: Suppose that all agents in $\mathcal{E}$ have the same utility function and thus the same demand function $\bar{x}$, with $\bar{x}(p,w) = \bar{x}(p,1)w$, i.e., demand is linear in income. Then Z obeys the weak axiom.

Proof: Denote the economy's aggregate endowment $\sum_{a \in A} \omega^a$ by $\bar{\omega}$. Then $X(p) = \sum_{a \in A} \bar{x}(p, p \cdot \omega^a)$

$$= \sum_{a \in A} \bar{x}(p,1)(p \cdot \omega^a)$$

$$= \bar{x}(p,1) \left[\sum_{a \in A}(p \cdot \omega^a)\right]$$

$$= \bar{x}(p,1) \left[p \cdot (\sum_{a \in A} \omega^a)\right]$$

$$= \bar{x}(p, p \cdot (\sum_{a \in A} \omega^a))$$

$$= \bar{x}(p, p \cdot \bar{\omega}).$$

Demand Aggregation: example 1

Theorem: Suppose that all agents in $\mathcal{E}$ have the same utility function and thus the same demand function $\bar{x}$, with $\bar{x}(p,w) = \bar{x}(p,1)w$, i.e., demand is linear in income. Then Z obeys the weak axiom.

Proof: Denote the economy's aggregate endowment $\sum_{a\in A}\omega^a$ by $\bar{\omega}$. Then $X(p) = \sum_{a\in A}\bar{x}(p, p\cdot\omega^a)$

$$= \sum_{a\in A}\bar{x}(p,1)(p\cdot\omega^a)$$

$$= \bar{x}(p,1)\left[\sum_{a\in A}(p\cdot\omega^a)\right]$$

$$= \bar{x}(p,1)\left[p\cdot\left(\sum_{a\in A}\omega^a\right)\right]$$

$$= \bar{x}(p, p\cdot(\sum_{a\in A}\omega^a))$$

$$= \bar{x}(p, p\cdot\bar{\omega}).$$

so economy's aggregate demand behaves like the demand of an agent with endowment $\bar{\omega}$.

Demand Aggregation: example 1

Theorem: Suppose that all agents in $\mathcal{E}$ have the same utility function and thus the same demand function $\bar{x}$, with $\bar{x}(p,w) = \bar{x}(p,1)w$, i.e., demand is linear in income. Then Z obeys the weak axiom.

Proof: Denote the economy's aggregate endowment $\sum_{a\in A}\omega^a$ by $\bar{\omega}$. Then $X(p) = \sum_{a\in A}\bar{x}(p, p\cdot\omega^a)$

$$= \sum_{a\in A}\bar{x}(p,1)(p\cdot\omega^a)$$

$$= \bar{x}(p,1)\left[\sum_{a\in A}(p\cdot\omega^a)\right]$$

$$= \bar{x}(p,1)\left[p\cdot\left(\sum_{a\in A}\omega^a\right)\right]$$

$$= \bar{x}\left(p, p\cdot\left(\sum_{a\in A}\omega^a\right)\right)$$

$$= \bar{x}(p, p\cdot\bar{\omega}).$$

so economy's aggregate demand behaves like the demand of an agent with endowment $\bar{\omega}$. Therefore, its excess demand function $Z(p) = \bar{x}(p, p\cdot\bar{\omega}) - \bar{\omega}$ obeys the weak axiom. **QED**

The law of demand

Let $\mathcal{O} \subseteq R^l$; $F : \mathcal{O} \to R^l$ is monotonic (obeys the law of demand) if

$$(p - p') \cdot (F(p) - F(p')) \leq 0 \text{ for any } p \text{ and } p' \text{ in } \mathcal{O}.$$

The law of demand

Let $\mathcal{O} \subseteq R^l$; $F : \mathcal{O} \to R^l$ is monotonic (obeys the law of demand) if

$$(p - p') \cdot (F(p) - F(p')) \leq 0 \text{ for any } p \text{ and } p' \text{ in } \mathcal{O}.$$

Suppose $p = (p_1, p_2, ..., p_l)$ and $p' = (p'_1, p_2, ..., p_l)$, then if F obeys the law of demand, we obtain

$$(p_1 - p'_1)(F_1(p) - F_1(p')) \leq 0.$$

The law of demand

Let $\mathcal{O} \subseteq R^l$; $F : \mathcal{O} \to R^l$ is monotonic (obeys the law of demand) if

$$(p - p') \cdot (F(p) - F(p')) \leq 0 \text{ for any } p \text{ and } p' \text{ in } \mathcal{O}.$$

Suppose $p = (p_1, p_2, ..., p_l)$ and $p' = (p'_1, p_2, ..., p_l)$, then if F obeys the law of demand, we obtain

$$(p_1 - p'_1)(F_1(p) - F_1(p')) \leq 0.$$

Proposition: Let $\mathcal{O}$ be an open and convex set in R^l and suppose that $F : \mathcal{O} \to R^l$ is a differentiable function defined on $\mathcal{O}$. Then F obeys the law of demand if and only if its derivative matrix $\partial_p F(p)$ is negative semidefinite for all $p \in \mathcal{O}$.

The law of demand

Let $\mathcal{O} \subseteq R^l$; $F : \mathcal{O} \to R^l$ is monotonic (obeys the law of demand) if

$$(p - p') \cdot (F(p) - F(p')) \leq 0 \text{ for any } p \text{ and } p' \text{ in } \mathcal{O}.$$

Suppose $p = (p_1, p_2, ..., p_l)$ and $p' = (p'_1, p_2, ..., p_l)$, then if F obeys the law of demand, we obtain

$$(p_1 - p'_1)(F_1(p) - F_1(p')) \leq 0.$$

Proposition: Let $\mathcal{O}$ be an open and convex set in R^l and suppose that $F : \mathcal{O} \to R^l$ is a differentiable function defined on $\mathcal{O}$. Then F obeys the law of demand if and only if its derivative matrix $\partial_p F(p)$ is negative semidefinite for all $p \in \mathcal{O}$.

The (Marshallian) demand function $\bar{x}$ does not generally obey the law of demand, but the law of demand generally holds for the Hicksian demand function.

The law of demand

Given a price vector $p \gg 0$ and a utility function u, consider the following problem:

($\star$) minimize $p \cdot x$ subject to $u(x) \geq \hat{u}$.

Suppose, that u is continuous, monotone, and strictly quasiconcave.

Then, so long as $\hat{u}$ is in the range of u, the problem ($\star$) admits a *unique* solution x^*, with $u(x^*) = \hat{u}$.

The law of demand

Given a price vector $p \gg 0$ and a utility function u, consider the following problem:

($\star$) minimize $p \cdot x$ subject to $u(x) \geq \hat{u}$.

Suppose, that u is continuous, monotone, and strictly quasiconcave.

Then, so long as $\hat{u}$ is in the range of u, the problem ($\star$) admits a *unique* solution x^*, with $u(x^*) = \hat{u}$.

Denote the range of u by $\mathcal{U}$, i.e., $\mathcal{U} = \{u(x) : x \in R^l_+\}$.
For any $(p, u) \in R^l_{++} \times \mathcal{U}$, we denote the solution to ($\star$) by $h(p, u)$.

The function h is known as the Hicksian demand function.

The law of demand

Given a price vector $p \gg 0$ and a utility function u, consider the following problem:

($\star$) minimize $p \cdot x$ subject to $u(x) \geq \hat{u}$.

Suppose, that u is continuous, monotone, and strictly quasiconcave.

Then, so long as $\hat{u}$ is in the range of u, the problem ($\star$) admits a *unique* solution x^*, with $u(x^*) = \hat{u}$.

Denote the range of u by $\mathcal{U}$, i.e., $\mathcal{U} = \{u(x) : x \in R^l_+\}$.
For any $(p, u) \in R^l_{++} \times \mathcal{U}$, we denote the solution to ($\star$) by $h(p, u)$.

The function h is known as the Hicksian demand function.

The map from (p, u) to $e(p, w) = p \cdot h(p, u)$ is called the expenditure function.

The law of demand

Proposition: h obeys the law of demand.

Proof: By definition of h, $p' \cdot h(p', u) \leq p' \cdot h(p, u)$ and $p \cdot h(p, u) \leq p \cdot h(p', u)$.

The law of demand

Proposition: h obeys the law of demand.

Proof: By definition of h, $p' \cdot h(p', u) \leq p' \cdot h(p, u)$ and $p \cdot h(p, u) \leq p \cdot h(p', u)$. This may re-written as

$$-p' \cdot (h(p, u) - h(p', u)) \leq 0 \text{ and}$$

$$p \cdot (h(p, u) - h(p', u)) \leq 0.$$

Summing these inequalities we obtain

$$(p - p') \cdot (h(p, u) - h(p', u)) \leq 0.$$

. **QED**

The law of demand

Proposition: h obeys the law of demand.

Proof: By definition of h, $p' \cdot h(p', u) \leq p' \cdot h(p, u)$ and $p \cdot h(p, u) \leq p \cdot h(p', u)$. This may re-written as

$$-p' \cdot (h(p, u) - h(p', u)) \leq 0 \text{ and}$$

$$p \cdot (h(p, u) - h(p', u)) \leq 0.$$

Summing these inequalities we obtain

$$(p - p') \cdot (h(p, u) - h(p', u)) \leq 0.$$

. **QED**

The Marshallian demand $\bar{x}$ obeys the law of demand if

$$(p - p') \cdot (\bar{x}(p, w) - \bar{x}(p', w)) \leq 0$$

at any two price vectors p and p', and at any fixed income w.

The law of demand

The Marshallian demand function, $\bar{x}$, obeys that law of demand if and only if the derivative matrix $\partial_p \bar{x}(p, w)$ is negative semidefinite.

By the Slutsky decomposition,

$$\partial_p \bar{x}(p, w) = S(p, w) - I(p, w),$$

where $S(p, w)$ and $I(p, w)$ are the substitution and income effect matrices.

Recall that $S(p, w) = \partial_p h(p, v(p, w))$, where $v(p, w) \equiv u(\bar{x}(p, w))$ is the indirect utility function.

S is negative semidefinite since h obeys the law of demand.

But I may not be positive semidefinite; hence the negative semidefiniteness of $\partial_p \bar{x}(p, w)$ is not guaranteed.

The law of demand

Proposition: I is positive semidefinite, and hence $\bar{x}$ obeys the law of demand, if $\bar{x}$ is linear in income.

The law of demand

Proposition: I is positive semidefinite, and hence $\bar{x}$ obeys the law of demand, if $\bar{x}$ is linear in income.

Proof: The ij entry of the matrix $I(p,w)$ is

$$\frac{\partial \bar{x}_j}{\partial w}(p,w)\,\bar{x}_i(p,w).$$

So $I(p,w) = B^T A$, where $B = (\bar{x}_1(p,w), \bar{x}_2(p,w), ..., \bar{x}_l(p,w))$ and

$$A = \left(\frac{\partial \bar{x}_1}{\partial w}(p,w), \frac{\partial \bar{x}_2}{\partial w}(p,w), ..., \frac{\partial \bar{x}_l}{\partial w}(p,w)\right).$$

If $\bar{x}(p,w) = \bar{x}(p,1)w$, then $\frac{\partial \bar{x}_i}{\partial w}(p,w) = \bar{x}_i(p,1)$. For any column vector $v \in R^l$,

$$v \cdot I(p,w)v = w\left(v \cdot \bar{x}(p,1)\right)^2 \geq 0.$$

Since $S(p,w)$ is negative semidefinite, we conclude that $\partial_p \bar{x}(p,w)$, which equals $S(p,w) - I(p,w)$ is a negative semidefinite matrix. **QED**

The law of demand

Theorem: (Milleron, Mitjuschin-Polterovich) Suppose u is increasing, strictly quasiconcave, and concave. Then u generates a demand function $\bar{x}$ obeying the law of demand if

$$-\frac{x \cdot \partial^2 u(x) x}{\partial u(x) x} \leq 4.$$

If the inequality is strict, then we obtain the *strict* law of demand, i.e.,

$$(p - p') \cdot (\bar{x}(p,w) - \bar{x}(p,w)) < 0 \text{ whenever } p \neq p'.$$

The law of demand

Theorem: (Milleron, Mitjuschin-Polterovich) Suppose u is increasing, strictly quasiconcave, and concave. Then u generates a demand function $\bar{x}$ obeying the law of demand if

$$-\frac{x \cdot \partial^2 u(x) x}{\partial u(x) x} \leq 4.$$

If the inequality is strict, then we obtain the *strict* law of demand, i.e.,

$$(p - p') \cdot (\bar{x}(p, w) - \bar{x}(p, w)) < 0 \text{ whenever } p \neq p'.$$

Corollary: Suppose $u(x) = \sum_{i=1}^{l} v_i(x_i)$. Then its demand function obeys the law of demand if, for all i,

$$-\frac{t v_i''(t)}{v_i'(t)} \leq 4.$$

Demand Aggregation

Why the fuss about the law of demand?

Observation 1: If a function F obeys the strict law of demand then it obeys the weak axiom.

Demand Aggregation

Why the fuss about the law of demand?

Observation 1: If a function F obeys the strict law of demand then it obeys the weak axiom.

Observation 2: The law of demand is preserved by aggregation.

Suppose

$$F(p) = \sum_{a \in A} \bar{x}^a(p, w^a),$$

i.e., $F(p)$ is market demand at price p. Then if $\bar{x}^a$ obeys the (strict) law of demand for all $a \in A$, F also obeys the (strict) law of demand.

Hence F obeys the weak axiom if $\bar{x}^a$ obeys the strict law of demand (for all agents a).

Demand Aggregation: example 2

Consider an exchange economy with collinear endowments, i..e, $\omega^a = t^a \bar{\omega}$, with $t^a > 0$ and $\sum_{a \in A} t^a = 1$.

Suppose that $\bar{x}^a$ obeys the (strict) law of demand. Consider two distinct prices p and p' with $p \cdot \bar{\omega} = p' \cdot \bar{\omega}$. Then

$$(p - p') \cdot [\bar{x}^a(p, t^a p \cdot \bar{\omega}) - \bar{x}^a(p', t^a p' \cdot \bar{\omega})] \leq (<)\, 0.$$

Demand Aggregation: example 2

Consider an exchange economy with collinear endowments, i..e, $\omega^a = t^a \bar{\omega}$, with $t^a > 0$ and $\sum_{a \in A} t^a = 1$.

Suppose that $\bar{x}^a$ obeys the (strict) law of demand. Consider two distinct prices p and p' with $p \cdot \bar{\omega} = p' \cdot \bar{\omega}$. Then

$$(p - p') \cdot [\bar{x}^a(p, t^a p \cdot \bar{\omega}) - \bar{x}^a(p', t^a p' \cdot \bar{\omega})] \leq (<)\, 0.$$

Summing across all agents we obtain

$$(p - p') \cdot (X(p) - X(p')) \leq (<)\, 0$$

and thus Z obeys the restricted (strict) law of demand:

$$(p - p') \cdot (Z(p) - Z(p')) \leq (<)\, 0 \text{ if } p \neq p' \text{ and } p \cdot \bar{\omega} = p' \cdot \bar{\omega}.$$

Demand Aggregation: example 2

Proposition: Suppose Z obeys the restricted strict law of demand. Then Z obeys the weak axiom.

Demand Aggregation: example 2

Proposition: Suppose Z obeys the restricted strict law of demand. Then Z obeys the weak axiom.

Proof: Recall that Z obeys the weak axiom if at any p and p', with $Z(p) \neq Z(p')$,

$$p' \cdot Z(p) \leq 0 \Longrightarrow p \cdot Z(p') > 0.$$

Given p and p', choose λ such that $p'' = \lambda p'$ satisfies $p \cdot \bar{\omega} = p'' \cdot \bar{\omega}$. Then, since Z obeys the restricted strict law of demand,

$$(p - p'') \cdot (Z(p) - Z(p'')) < 0.$$

This can re-written as $-p'' \cdot Z(p) - p \cdot Z(p'') < 0$.

Demand Aggregation: example 2

Proposition: Suppose Z obeys the restricted strict law of demand. Then Z obeys the weak axiom.

Proof: Recall that Z obeys the weak axiom if at any p and p', with $Z(p) \neq Z(p')$,

$$p' \cdot Z(p) \leq 0 \Longrightarrow p \cdot Z(p') > 0.$$

Given p and p', choose λ such that $p'' = \lambda p'$ satisfies $p \cdot \bar{\omega} = p'' \cdot \bar{\omega}$. Then, since Z obeys the restricted strict law of demand,

$$(p - p'') \cdot (Z(p) - Z(p'')) < 0.$$

This can re-written as $-p'' \cdot Z(p) - p \cdot Z(p'') < 0$.

Suppose $p' \cdot Z(p) \leq 0$; then $p'' \cdot Z(p) \leq 0$. This implies that $p \cdot Z(p'') > 0$.
Since Z is zero-homogeneous, we obtain $p \cdot Z(p') > 0$. **QED**