Lectures on General Equilibrium Theory

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Uniqueness and Stability of Equilibrium

John Quah

john.quah@economics.ox.ac.uk

Generic uniqueness of equilibrium

Recall: Z obeys WARP if at prices p and p' with $Z(p) \neq Z(p')$,

$$p' \cdot Z(p) \leq 0 \Longrightarrow p \cdot Z(p') > 0.$$

Proposition: Suppose that Z obeys the weak axiom. Then the set of equilibrium prices form a convex set.

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Proposition: Suppose that Z obeys the weak axiom. Then the set of equilibrium prices form a convex set.

Proof: Let p^* and p^{**} be two equilibrium prices. Suppose $\bar{p} = tp^* + (1-t)p^{**}$ is *not* an equilibrium price, i.e., $Z(\bar{p}) \neq 0$.

Since $\bar{p} \cdot Z(p^*) = 0$, the weak axiom implies that $p^* \cdot Z(\bar{p}) > 0$. By a similar argument, $p^{**} \cdot Z(\bar{p}) > 0$. So

$$\bar{p} \cdot Z(\bar{p}) = tp^* \cdot Z(\bar{p}) + (1-t)p^{**} \cdot Z(\bar{p}) > 0,$$

which contradicts Walras' Law. **QED**

Generic uniqueness of equilibrium

But non-singleton convex equilibrium sets are non-generic. So when Z obeys the WA, generically, the price equilibrium is *unique*.

Proposition: Suppose Z obeys the weak axiom and there is a unique p^* such that $Z(p^*) = 0$. Then Z has the following property, which we call the weak axiom at equilibrium: for any p not collinear with p^*,

$$(p - p^*) \cdot Z(p) < 0.$$

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Proof: Note that $Z(p) \neq Z(p^*) = 0$. Furthermore, $p \cdot Z(p^*) = 0$. By the weak axiom, $p^* \cdot Z(p) > 0$, so

$$(p - p^*) \cdot Z(p) = -p^* \cdot Z(p) < 0.$$

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Interpretation of WAE: if $p = (p_1, p_2^*, p_3^*, ..., p_l^*)$, then

$$(p - p^*) \cdot Z(p) = (p_1 - p_1^*) Z_1(p) < 0.$$

When price of 1 is higher than its equilibrium price, there is excess supply of 1; etc.

Walras' tatonnement

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Not generally. Depends on Z.

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Lemma: Let $p(t)$ be the solution to Walras tatonnement at the initial condition $p(0) = \bar{p}$. Then

$$\sum_{i=1}^{l} p_i^2(t) = \sum_{i=1}^{l} \bar{p}_i^2 \text{ for all } t.$$

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Proof: Note that

$$\frac{d}{dt}\left(\sum_{i=1}^{l} p_i^2(t)\right) = 2\left(\sum_{i=1}^{l} p_i(t)\frac{dp_i}{dt}(t)\right) = 2\left(\sum_{i=1}^{l} p_i(t) Z_i(p(t))\right).$$

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Thus, the solution $p(t)$ lies on the surface of a higher dimensional sphere with radius $\sqrt{\sum_{i=1}^{l} \bar{p}_i{}^2}$.

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Consider the Lyapunov function $L(p) = \sum_{i=1}^{l}(p_i - p_i^*)^2$. Then

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$$\frac{dL}{dt} = 2 \sum_{i=1}^{l} (p_i - p_i^*) \frac{dp_i}{dt}.$$

The latter equals $2(p - p^*) \cdot Z(p)$, so $\frac{dL}{dt} < 0$. **QED**