Lectures on General Equilibrium Theory

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Revealed preference tests of utility maximization and general equilibrium

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Observable Restrictions

Let $\mathcal{O} = \{(p_t, x_t)\}_{t \in T}$ be a set of observations.

$x_t \in R^l_+$ is the observed demand at price $p_t \in R^l_{++}$.

We know that certain observations are *not* consistent with utility-maximization with increasing utility functions.

In other words, the utility-maximization hypothesis leads to observable restrictions on a data set.

What restrictions on the data are sufficient for us to recover a utility function generating the data?

Generalized Axiom of Revealed Preference

The set $\mathcal{O} = \{(p_t, x_t)\}_{t \in T}$ satisfies the generalized axiom of revealed preference (GARP) if whenever there are observations satisfying ($\star$):

$$
\begin{aligned}
p_{t^1} \cdot x_{t^2} &\leq p_{t^1} \cdot x_{t^1} \\
p^{t^2} \cdot x_{t^3} &\leq p_{t^2} \cdot x_{t^2} \\
&. \\
&. \\
p_{t^{n-1}} \cdot x_{t^n} &\leq p_{t^{n-1}} \cdot x_{t^{n-1}} \\
p_{t^n} \cdot x_{t^1} &\leq p_{t^n} \cdot x_{t^n},
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Generalized Axiom of Revealed Preference

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Proof: Suppose that $\mathcal{O}$ is drawn from a consumer maximizing the utility function U.

If $p_s \cdot x_r \leq p_s \cdot x_s$ then $U(x_r) \leq U(x_s)$. And since U is increasing, if $p_s \cdot x_r < p_s \cdot x_r$ then $U(x_r) < U(x_s)$.

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The displayed inequalities in ($\star$) imply that

$$U(x_{t_1}) \leq U(x_{t_n}) \leq U(x_{t_{n-1}}) \leq \ldots \leq U(x_{t_3}) \leq U(x_{t_2}) \leq U(x_{t_1}). \qquad (2)$$

Clearly we obtain a contradiction if any inequality is strict. So they must all be equalities, which mean that the observations satisfying ($\star$) must also be equalities. **QED**

Afriat's Theorem

Theorem: (Afriat) Let $\mathcal{O} = \{p_t, x_t\}_{t \in T}$ be a set of observations. The following statements about $\mathcal{O}$ are equivalent:

(1) There is an increasing and concave utility function U such that

$$x_s \in \text{argmax}_{x \in B(p_s, p_s \cdot x_s)} U(x).$$

(2) $\mathcal{O}$ obeys GARP.

(3) There are numbers ϕ_s and λ_s, with $\lambda_s > 0$ (associated to each observation $s \in T$) such that

$$\phi_s \leq \phi_k + \lambda_k p_k \cdot (x_s - x_k) \text{ for any observations } s \text{ and } k. \qquad (3)$$

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The problem of finding (ϕ_s, λ_s) (for each observation s) that solve (**??**) is a linear programming problem.

Such problems are known to be solvable in the sense that there is an algorithm to determine in a finite number of steps, whether or not the system of linear inequalities given by (3) has a solution.

Afriat's Theorem

Whenever U is increasing, we know that $\mathcal{O}$ obeys GARP.

Afriat's Theorem also say that if GARP holds, then there is a utility function generating $\mathcal{O}$ that is increasing *and concave*.

Hence, if a data set is consistent with an increasing utility function, it must be consistent with an increasing and concave utility function.

In fact, the utility function constructed from the data is

$$U(x) = \min_{(p_t, x_t) \in \mathcal{O}} \left\{ \phi_t + \lambda_t p_t \cdot (x - x_t) \right\}.$$

This is always a concave function, and it is an increasing utility function since $\lambda_t > 0$ for all t.

Afriat's Theorem

Proof of Afriat's Theorem: The first proposition says that (1) implies (2).

Recall: (3) There are numbers ϕ_s and λ_s, with $\lambda_s > 0$, such that

$$\phi_s \leq \phi_k + \lambda_k p_k \cdot (x_s - x_k) \text{ for any observations } s \text{ and } k.$$

To see that (3) implies (1) consider

$$U(x) = \min_{(p_t, x_t) \in \mathcal{O}} \{\phi_t + \lambda_t p_t \cdot (x - x_t)\}.$$

This is an increasing utility function since $\lambda_t > 0$ for all t. It also generates the observations in $\mathcal{O}$.

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Let x satisfy $p_s \cdot x = p_s \cdot x_s$, i.e., x is on the budget plane of the agent at price p_s and income $p_s \cdot x_s$.

It follows from the definition of U that $U(x) \leq \phi_s$.

To guarantee that x_s maximizes utility in $B(p_s, p_s \cdot x_s)$ it suffices to show that $U(x_s) = \phi_s$. This follows immediately from (3). **QED**

Afriat's Theorem

Proof that (2) implies (3): To simplify, assume that observations are generic, i.e., for any $t' \neq t$, $p_t \cdot x_{t'} \neq p_t \cdot x_t$.

In other words, either $p_t \cdot x_{t'} < p_t \cdot x_t$ or $p_t \cdot x_{t'} > p_t \cdot x_t$.

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Denote the set of observed demands by $\mathcal{X}$, i.e., $\mathcal{X} = \{x_t\}_{t \in T}$.

The order $\succ^{**}$ on $\mathcal{X}$ is defined as follows: $x^t \succ^{**} x_s$ if $p_t \cdot x_s < p_t \cdot x_t$. In this case we say that x^t is directly revealed preferred to x^s.

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The order $\succ^*$ is the transitive closure of $\succ^{**}$, i.e., $x_t \succ^* x_s$ if there are observations t^1, t^2,..., t^n such that

$$x_t \succ^{**} x_{t_1} \succ^{**} x_{t_2} \succ^{**} ... \succ^{**} x_{t_n} \succ^{**} x_s.$$

If $x_t \succ^* x_s$ we say that x_t is revealed preferred to x_s.

We refer to $\succ^*$ as the revealed preference order on $\mathcal{X}$ derived from $\mathcal{O}$.

Afriat's Theorem

Lemma: The order $\succ^*$ is transitive and irreflexive.

An order $\succ$ on $\mathcal{X}$ is an *extension* of another order $\succ^*$ if $x_t \succ x_s$ whenever $x_t \succ^* x_s$.

Lemma: There is an extension $\succ$ of $\succ^*$ that is transitive, irreflexive, and complete, i.e., for any two elements x_t and x_s in $\mathcal{X}$, either $x_t \succ x_s$ or $x_s \succ x_t$.

Note: There could be more than one such extension.

Afriat's Theorem

Lemma: Let $\succ$ be a complete extension of $\succ^*$. Assume that $\mathcal{X} = \{x_1, x_2, ..., x_N\}$, with

$$x_1 \prec x_2 \prec x_3 \prec ... \prec x_N.$$

Then there are numbers ϕ_s and λ_s, with $\lambda_s > 0$, such that

$$\phi_s < \phi_k + \lambda_k p_k \cdot (x_s - x_k) \text{ for all } k \neq s$$

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Proof: Denote $p_i \cdot (x_j - x_i)$ by a_{ij}. Choose ϕ_1 to be any number and λ_1 to be any positive number. Since $x_j \succ x_1$ for all $j > 1$,

$$a_{1j} = p_1 \cdot (x_j - x_1) > 0.$$

So there is ϕ_2 such that

$$\phi_1 < \phi_2 < \min_{j>1}\{\phi_1 + \lambda_1 a_{1j}\}.$$

Now choose $\lambda_2 > 0$ sufficiently small so that

$$\phi_1 < \phi_2 + \lambda_2 a_{21}.$$

Since $a_{2j} = p_2 \cdot (x_j - x_2) > 0$ for $j > 2$, we have $\phi_2 < \min_{j>2}\{\phi_2 + \lambda_2 a_{2j}\}$. Therefore, we can choose ϕ_3 such that

$$\phi_2 < \phi_3 < \min\left\{\min_{j>2}\{\phi_2 + \lambda_2 a_{2j}\}, \min_{j>1}\{\phi_1 + \lambda_1 a_{1j}\}\right\}.$$

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Therefore, we can choose ϕ_3 such that

$$\phi_2 < \phi_3 < \min\left\{\min_{j>2}\{\phi_2 + \lambda_2 a_{2j}\}, \min_{j>1}\{\phi_1 + \lambda_1 a_{1j}\}\right\}.$$

We can choose $\lambda_3 > 0$ and sufficiently small such that

$$\phi_i < \phi_3 + \lambda_3 a_{3i} \text{ for } i = 1, 2.$$

More generally, we can choose ϕ_k such that

$$\phi_{k-1} < \phi_k < \min_{s \le k-1}\left\{\min_{j>s}\{\phi_s + \lambda_s a_{sj}\}\right\}$$

and $\lambda_k > 0$ and sufficiently small so that

$$\phi_i < \phi_k + \lambda_k a_{ki} \text{ for } i \le k-1.$$

In this way, we obtain

$$\phi_s < \phi_k + \lambda_k a_{ks} \text{ for all } k \ne s,$$

as required. **QED**

Brown-Matzkin Theorem

Consider an observer who makes several observations of the outcome of an economy.

Each observation consists of a price vector p_t, aggregate endowment $\bar{\omega}_t$, and the income distribution $\{w_t^a\}_{a \in A}$.

With no loss of generality, assume that $\sum_{a \in A} w_t^a = p_t \cdot \bar{\omega}_t = 1$.

We denote the set of observations by $\mathcal{O} = \{(p_t, \omega_t, \{w_t^a\}_{a \in A})\}_{t \in T}$.

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Note that individual demands are not observed.

When can we find increasing utility functions U^a, generating the demand function $\bar{x}^a$, such that

$$\sum_{a \in A} \bar{x}^a(p_t, w_t^a) = \bar{\omega}_t \text{ for every observation } t?$$

If this is possible, we say that $\mathcal{O}$ is Walrasian rationalizable.

Brown-Matzkin Theorem

Suppose that $\mathcal{O} = \{(p_t, \omega_t, \{w_t^a\}_{a \in A})\}_{t \in T}$ is Walrasian rationalizable, so we can find increasing utility functions U^a, generating the demand function $\bar{x}^a$, such that

$$\sum_{a \in A} \bar{x}^a(p_t, w_t^a) = \bar{\omega}_t \text{ for every observation } t.$$

Since individual endowments are not observed, each observation can be thought of as arising from an exchange economy where agent a has utility function U^a and endowment $w_t^a \bar{\omega}_t$.

The data is generated by changes to endowments, while the utility function of each agent is assumed to be unchanged across observations.

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Theorem: (Brown-Matzkin) Given any set of observations $\mathcal{O}$, there is an algorithm that could determine in a finite number of steps whether or not the set of observations is Walrasian rationalizable.

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Proof: By Afriat's Theorem, $\mathcal{O}$ is Walrasian rationalizable if and only if there is $x_t^a \geq 0$ such that

(1) $\sum_{a \in A} x_t^a = \bar{\omega}_t$ for all t;
(2) $p_t \cdot x_t^a = w_t^a$ for all a and t; and
(3) the set $\{p^t, x_t^a\}_{t \in T}$ obeys GARP for all a.

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By Afriat's Theorem again, this is equivalent to finding (for every t and a) $x_t^a \geq 0$, ϕ_t^a, and $\lambda_t^a > 0$, such that

(1) $\sum_{a \in A} x_t^a = \bar{\omega}_t$ for all t;
(2) $p_t \cdot x_t^a = w_t^a$ for all a and t; and
(3) $\phi_s^a \leq \phi_k^a + \lambda_k^a p_k \cdot (x_s^a - x_k^a)$ for all a, k and s.

This is a system of quadratic inequalities. It is known that these are solvable (Tarski-Seidenberg Theorem). **QED**