Lectures on
General Equilibrium Theory

Equilibrium in a financial economy

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Financial Assets

Assume that there are two dates, Today (date 0) and Tomorrow (date 1). There are $L$ states of the world tomorrow.

Security/Asset is a promise of payment (positive or negative), conditional on the realization of the state.

We write payoff of security $s$ as a column vector, called the payoff vector of security $s$:

$$
\begin{pmatrix}
  d_{1s} \\
  d_{2s} \\
  d_{3s} \\
  \vdots \\
  d_{Ls}
\end{pmatrix}
$$
Payoff Matrix

If economy has $S$ securities (called 1, 2,..., $S$), then the payoff matrix is

$$D = \begin{pmatrix}
  d_{11} & d_{12} & \cdots & d_{1S} \\
  d_{21} & d_{22} & \cdots & d_{2S} \\
  d_{31} & d_{32} & \cdots & d_{3S} \\
  \vdots & \vdots & \ddots & \vdots \\
  d_{L1} & d_{L2} & \cdots & d_{LS}
\end{pmatrix}.$$ 

Trade in securities/portfolio represented by the column vector $z$ in $\mathbb{R}^S$.

If $z_s > 0$ then agent is buying asset $s$;
if $z_s < 0$ then agent is selling asset $s$. 

Lectures on General Equilibrium Theory

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Contingent consumption

A portfolio \( z \) (in \( R^S \)) gives a payoff in state \( i \) of

\[
z_1 d_{i1} + z_2 d_{i2} + ... z_S d_{iS}.
\]

This is the \( i \)th entry in the column vector \( Dz \) in \( R^L \) since

\[
Dz = z_1 \begin{pmatrix} d_{11} \\ d_{21} \\ . \\ d_{L1} \end{pmatrix} + z_2 \begin{pmatrix} d_{12} \\ d_{22} \\ . \\ d_{L2} \end{pmatrix} + ... z_S \begin{pmatrix} d_{1S} \\ d_{2S} \\ . \\ d_{LS} \end{pmatrix}.
\]

So \( Dz \) is the payoff vector of the portfolio \( z \).

The subspace \( \{ Dz ; z \in R^S \} \), denoted by \( \text{Span}(D) \), is called the span of the security payoffs or the asset span.
Incomplete Markets

If $\text{Span}(D)$ is a *strict* subspace of $R^L$, then the economy has incomplete markets.

Suppose agent has endowment $\omega = (\omega_0, \omega_1, ..., \omega_L) \in R^{L+1}_+$, where $\omega_0$ in $R_+$ is agent’s endowment at $t = 0$ and $\omega_{-0} = (\omega_1, \omega_2, ..., \omega_L)$ is his endowment at $t = 1$.

With a portfolio $\bar{z}$, the agent’s contingent consumption at $t = 1$ is $\omega_{-0} + D\bar{z}$.

With incomplete markets, there are contingent consumption bundles that the agent cannot achieve even if he could assemble any portfolio he likes, i.e.,

there is $x^* \in R^L_+$ such that there is no $z$ with $\omega_{-0} + Dz = x^*$. 
Incomplete Markets

Example: Suppose there are three states of the world at $t = 1$ and just two securities, 1 and 2, with payoff vectors $(1, 1, 0)$ and $(0, 1, 2)$ respectively.

An agent’s endowment at $t = 1$ is $\omega_0 = (2, 3, 0)$. The portfolio $z = (z_1, z_2)$ gives this agent contingent consumption of

$$\omega_0 + Dz = \begin{pmatrix} 2 \\ 3 \\ 0 \end{pmatrix} + z_1 \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + z_2 \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}.$$ 

Note that $M = \{\omega_0 + Dz : z \in R^2\}$ is a plane in $R^3$ passing through the point $\omega_0$.

In particular $R^3_+ \not\subseteq M$ – so market is incomplete.
Agent’s utility maximization problem

Agent has endowment \( \omega = (\omega_0, \omega_{-0}) \) in \( \mathbb{R}^{L+1}_+ \).
Agent’s utility maximization problem

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Agent has utility function \( U : \mathbb{R}^{L+1}_+ \rightarrow \mathbb{R} \).
Agent’s utility maximization problem

Agent has endowment \( \omega = (\omega_0, \omega_{-0}) \) in \( \mathbb{R}_+^{L+1} \).

Agent has utility function \( U : \mathbb{R}_+^{L+1} \rightarrow \mathbb{R} \). For example,

\[
U(x_0, x_{-0}) = u(x_0) + \delta \left[ \pi_1 v(x_1) + \pi_2 v(x_2) + \ldots + \pi_L v(x_L) \right].
\]
Agent’s utility maximization problem

Agent has endowment $\omega = (\omega_0, \omega_0)$ in $\mathbb{R}^{L+1}_+$. Agent has utility function $U : \mathbb{R}^{L+1}_+ \to \mathbb{R}$. For example,

$$U(x_0, x_0) = u(x_0) + \delta [\pi_1 v(x_1) + \pi_2 v(x_2) + \ldots + \pi_L v(x_L)].$$

Suppose $q \in \mathbb{R}^s$ is the asset price vector. Then the agent’s budget set is

$$B(q, \omega, D) = \left\{ x \in \mathbb{R}^{L+1}_+ : \begin{array}{l} x_0 \leq \omega_0 - q \cdot z \\ x_0 - 0 \leq \omega_0 + Dz \end{array} \right\}.$$ 

Note that the price of Today’s consumption is normalized at Today 1.
Agent’s utility maximization problem

Agent maximizes $U(x)$ subject to $x$ in $B(q, \omega, D)$.

Then $\hat{x} \in \text{argmax}_{x \in B(q, \omega, D)} U(x)$ is the agent’s demand for contingent consumption and $\hat{z}$ such that

\[
\begin{align*}
\hat{x}_0 & \leq \omega_0 - q \cdot \hat{z} \quad \text{and} \\
\hat{x}_{-0} & \leq \omega_{-0} + D \hat{z}
\end{align*}
\]

is the agent’s demand for securities.

Note: if $U$ is strictly monotone (obeys (P2)), then the inequalities above are equalities.
Example

Agent’s utility is $U(x_0, x_1, x_2) = \ln x_0 + \frac{1}{2} \ln x_1 + \frac{1}{2} \ln x_2$

Two securities with payoff vectors $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ respectively.

Price is $q = (1, 2)$. Endowment $\omega = (3, 0, 0)$. 
Example

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Price is \( q = (1, 2) \). Endowment \( \omega = (3, 0, 0) \).

Budget Set = \( \left\{ x \in \mathbb{R}^3_+ : \begin{align*} x_0 &\leq 3 - z_1 - 2z_2 \\ x_1 &\leq z_1 + z_2 \\ x_2 &\leq z_2 \text{ for some } z = (z_1, z_2) \end{align*} \right\} \).
Example

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Two securities with payoff vectors $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ respectively.

Price is $q = (1, 2)$. Endowment $\omega = (3, 0, 0)$.

Budget Set $= \left\{ x \in \mathbb{R}_+^3 : \begin{array}{l} x_0 \leq 3 - z_1 - 2z_2 \\ x_1 \leq z_1 + z_2 \\ x_2 \leq z_2 \text{ for some } z = (z_1, z_2) \end{array} \right\}$.

Maximize $\ln(3 - z_1 - 2z_2) + \frac{1}{2} \ln(z_1 + z_2) + \frac{1}{2} \ln z_2$. 

Agent’s utility is $U(x_0, x_1, x_2) = \ln x_0 + \frac{1}{2} \ln x_1 + \frac{1}{2} \ln x_2$

Two securities with payoff vectors $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ respectively.

Price is $q = (1, 2)$. Endowment $\omega = (3, 0, 0)$.

Budget Set $= \left\{ x \in \mathbb{R}^3_+ : \begin{array}{l} x_0 \leq 3 - z_1 - 2z_2 \\ x_1 \leq z_1 + z_2 \\ x_2 \leq z_2 \end{array} \right\}$.

Maximize $\ln(3 - z_1 - 2z_2) + \frac{1}{2} \ln(z_1 + z_2) + \frac{1}{2} \ln z_2$.

Solution: $\bar{z}_1 = 0, \bar{z}_2 = \frac{3}{4}$.
Example

Agent’s utility is \( U(x_0, x_1, x_2) = \ln x_0 + \frac{1}{2} \ln x_1 + \frac{1}{2} \ln x_2 \)

Two securities with payoff vectors \( \begin{pmatrix} 1 \\ 0 \end{pmatrix} \) and \( \begin{pmatrix} 1 \\ 1 \end{pmatrix} \) respectively.

Price is \( q = (1, 2) \). Endowment \( \omega = (3, 0, 0) \).

\[
\left\{ x \in \mathbb{R}^3_+ : \begin{array}{l}
x_0 \leq 3 - z_1 - 2z_2 \\
x_1 \leq z_1 + z_2 \\
x_2 \leq z_2 \text{ for some } z = (z_1, z_2) \end{array} \right\}.
\]

Maximize \( \ln(3 - z_1 - 2z_2) + \frac{1}{2} \ln(z_1 + z_2) + \frac{1}{2} \ln z_2 \).

Solution: \( \bar{z}_1 = 0, \bar{z}_2 = \frac{3}{4}, \bar{x}_0 = \frac{3}{2}, \bar{x}_1 = \frac{3}{4}, \bar{x}_2 = \frac{3}{4} \).
The Financial Economy

A financial economy $\mathcal{F}$ consists of a payoff matrix $D$ and a set $A$ of agents, each of whom has an endowment $\omega^a = (\omega^a_0, \omega^a_{-0})$ in $R^{L+1}_+$ and a utility function $U^a : R^{L+1}_+ \rightarrow R$.

A price $q^*$ in $R^S$ is an equilibrium price of this financial economy if
The Financial Economy

A financial economy $\mathcal{F}$ consists of a payoff matrix $D$ and a set $A$ of agents, each of whom has an endowment $\omega^a = (\omega_0^a, \omega_{-0}^a)$ in $R^{L+1}_+$ and a utility function $U^a : R^{L+1}_+ \to R$.

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(i) for each agent $a$, there is $\hat{z}^a$ such that $D\hat{z}^a$ maximizes $U^a(x)$ in $B(q^*, \omega^a, D)$ and
The Financial Economy

A financial economy $\mathcal{F}$ consists of a payoff matrix $D$ and a set $A$ of agents, each of whom has an endowment $\omega^a = (\omega^a_0, \omega^a_{-0})$ in $R_+^{L+1}$ and a utility function $U^a : R_+^{L+1} \rightarrow R$.

A price $q^*$ in $R^S$ is an equilibrium price of this financial economy if

(i) for each agent $a$, there is $\hat{z}^a$ such that $D\hat{z}^a$ maximizes $U^a(x)$ in $B(q^*, \omega^a, D)$ and

(ii) $\sum_{a \in A} \hat{z}^a = 0$. 

Lectures on General Equilibrium Theory

* * * * *Equilibrium in a financial economy* – p. 10/25
Proposition: Suppose that $q^*$ is an equilibrium price of $\mathcal{F}$ and let $\hat{x}^a$ and $\hat{z}^a$ be agent $a$’s asset and consumption demand respectively. Then, provided $U^a$ obeys (P2),

$$
\sum_{a \in A} \hat{x}^a = \sum_{a \in A} \omega^a.
$$
Equilibrium in a financial economy

Proposition: Suppose that \( q^* \) is an equilibrium price of \( \mathcal{F} \) and let \( \hat{z}^a \) and \( \hat{x}^a \) be agent \( a \)'s asset and consumption demand respectively. Then, provided \( \U^a \) obeys (P2),

\[
\sum_{a \in A} \hat{x}^a = \sum_{a \in A} \omega^a.
\]

Proof: For agent \( a \), we have \( \hat{x}_0^a = \omega_0 - q^* \cdot \hat{z}^a \). Sum across \( a \), we obtain

\[
\sum_{a \in A} \hat{x}_0^a = \sum_{a \in A} \omega_0^a.
\]
Proposition: Suppose that \( q^* \) is an equilibrium price of \( F \) and let \( \hat{z}^a \) and \( \hat{x}^a \) be agent \( a \)'s asset and consumption demand respectively. Then, provided \( U^a \) obeys (P2),

\[
\sum_{a \in A} \hat{x}^a = \sum_{a \in A} \omega^a.
\]

Proof: For agent \( a \), we have \( \hat{x}^a_0 = \omega_0 - q^* \cdot \hat{z}^a \). Sum across \( a \), we obtain

\[
\sum_{a \in A} \hat{x}^a_0 = \sum_{a \in A} \omega^a_0.
\]

Since \( \hat{x}^a_{-0} = \hat{\omega}^a_{-0} + D \hat{z}^a \), summing across \( a \) gives us

\[
\sum_{a \in A} \hat{x}^a_{-0} = \sum_{a \in A} \omega^a_{-0}.
\]

\[\text{QED}\]
Constrained Pareto optimality

The allocation \( \{ x^a \}_{a \in A} \) is feasible if \( \sum_{a \in A} x^a = \sum_{a \in A} \omega^a \).

It is a constrained feasible allocation if it is feasible and there is \( \{ z^a \}_{a \in A} \) such that \( \sum_{a \in A} z^a = 0 \) and

\[
x^a_{-0} = \omega^a_{-0} + D z^a.
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Constrained Pareto optimality

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\[
x^a - \omega^a_0 + Dz^a.
\]

An allocation \( \{x^a\}_{a \in A} \) is constrained Pareto optimal if there does not exist a constrained feasible allocation \( \{\bar{x}^a\}_{a \in A} \) that is Pareto superior, i.e.,

\[
U^a(\bar{x}^a) \geq U^a(x^a)
\]

for all \( a \) and the inequality is strict for at least one agent.
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It is a constrained feasible allocation if it is feasible and there is \( \{z^a\}_{a \in A} \) such that \( \sum_{a \in A} z^a = 0 \) and

\[
x^a_0 = \omega^a_0 + Dz^a.
\]

An allocation \( \{x^a\}_{a \in A} \) is constrained Pareto optimal if there does not exist a constrained feasible allocation \( \{\bar{x}^a\}_{a \in A} \) that is Pareto superior, i.e.,

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\]

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**Theorem** [First welfare theorem for financial economies]: Suppose that \( U^a \) obeys (P2) for all \( a \). Then every equilibrium allocation is constrained Pareto optimal.
Constrained Pareto optimality

Proof: Let $q^*$ be the equilibrium price and $\{\hat{x}^a\}_{a \in A}$ the equilibrium allocation. Assume that $\{\bar{x}^a\}_{a \in A}$ is a Pareto superior constrained feasible allocation.
Constrained Pareto optimality

Proof: Let $q^*$ be the equilibrium price and $\{\hat{x}^a\}_{a \in A}$ the equilibrium allocation. Assume that $\{\bar{x}^a\}_{a \in A}$ is a Pareto superior constrained feasible allocation. For some agent $\tilde{a}$, we have $U^{\tilde{a}}(\bar{x}^{\tilde{a}}) > U^{\tilde{a}}(\hat{x}^{\tilde{a}})$. So the bundle $\bar{x}^{\tilde{a}}$ cannot be in $\tilde{a}$’s budget set - if it were, agent $\tilde{a}$ would have chosen this bundle instead of $\hat{x}^{\tilde{a}}$. 
Constrained Pareto optimality

Proof: Let $q^*$ be the equilibrium price and $\{\hat{x}^a\}_{a \in A}$ the equilibrium allocation. Assume that $\{\bar{x}^a\}_{a \in A}$ is a Pareto superior constrained feasible allocation.

For some agent $\tilde{a}$, we have $U^{\tilde{a}}(\bar{x}^{\tilde{a}}) > U^{\tilde{a}}(\hat{x}^{\tilde{a}})$.

So the bundle $\bar{x}^{\tilde{a}}$ cannot be in $\tilde{a}$’s budget set - if it were, agent $\tilde{a}$ would have chosen this bundle instead of $\hat{x}^{\tilde{a}}$. Thus

$$\bar{x}_0^{\tilde{a}} > \omega_0^{\tilde{a}} - q^* \cdot \tilde{z}^{\tilde{a}}.$$
Constrained Pareto optimality

Proof: Let $q^*$ be the equilibrium price and $\{\hat{x}^a\}_{a \in A}$ the equilibrium allocation. Assume that $\{\bar{x}^a\}_{a \in A}$ is a Pareto superior constrained feasible allocation.

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$$\bar{x}_0^{\tilde{a}} > \omega_0^{\tilde{a}} - q^* \cdot \tilde{z}^{\tilde{a}}.$$
Constrained Pareto optimality

Suppose $\bar{z}^a$ is the trade that achieves $\bar{x}^a$. We claim that

$$\bar{x}_0^a \geq \omega_0^a - q^* \cdot \bar{z}^a.$$
Constrained Pareto optimality

Suppose $\bar{z}^a$ is the trade that achieves $\bar{x}^a$. We claim that

$$\bar{x}_0^a \geq \omega_0^a - q^* \cdot \bar{z}^a.$$

Suppose not, then there is $\epsilon > 0$ such that $\bar{x}_0^a + \epsilon < \omega_0^a - q^* \cdot \bar{z}^a$.

So $\tilde{x}^a = \bar{x}^a + (\epsilon, 0, ..., 0)$ is in $B(q^*, \omega, D)$ and

$$U^a(\tilde{x}^a) > U^a(\bar{x}^a) \geq U^a(\hat{x}^a).$$

This cannot happen since $\hat{x}^a$ is $a$’s demand.
Constrained Pareto optimality

Suppose \( \tilde{z}^a \) is the trade that achieves \( \bar{x}^a \). We claim that

\[
\bar{x}_0^a \geq \omega_0^a - q^* \cdot \tilde{z}^a.
\]

Suppose not, then there is \( \epsilon > 0 \) such that

\[
\bar{x}_0^a + \epsilon < \omega_0^a - q^* \cdot \tilde{z}^a.
\]

So \( \tilde{x}^a = \bar{x}^a + (\epsilon, 0, \ldots, 0) \) is in \( B(q^*, \omega, D) \) and

\[
U^a(\tilde{x}^a) > U^a(\bar{x}^a) \geq U^a(\hat{x}^a).
\]

This cannot happen since \( \hat{x}^a \) is \( a \)'s demand.

Summing across \( a \), we obtain

\[
\sum_{a \in A} \bar{x}_0^a > \sum_{a \in A} \omega_0^a - q^* \cdot \left( \sum_{a \in A} \tilde{z}^a \right)
\]

\[
= \sum_{a \in A} \omega_0^a,
\]

which is a contradiction. 

QED
Invariance

Consider a financial economy and keep endowments and preferences fixed. Then a change in securities that leaves the span unchanged does not change the equilibrium in an essential way.
Invariance

Consider a financial economy and keep endowments and preferences fixed. Then a change in securities that leaves the span unchanged does not change the equilibrium in an essential way.

Lemma: Let \{1, 2, ..., S\} be a set of securities with no redundant securities and let $D$ be its payoff matrix. Let there be another set of $|S|$ securities, with payoff matrix $D'$ such that $\text{Span}(D)=\text{Span}(D')$.

Then there is invertible matrix $S \times S$ matrix $K$ such that $DK = D'$. 
Invariance

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**Lemma:** Let \( \{1, 2, \ldots, S\} \) be a set of securities with no redundant securities and let \( D \) be its payoff matrix. Let there be another set of \( |S| \) securities, with payoff matrix \( D' \) such that \( \text{Span} (D) = \text{Span} (D') \).

Then there is invertible matrix \( S \times S \) matrix \( K \) such that \( DK = D' \).

**Proof:** Exercise!
Example: Suppose $D = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $D' = \begin{pmatrix} 2 & 0 \\ 1 & 1 \end{pmatrix}$.

\[
\begin{pmatrix} 2 \\ 1 \end{pmatrix} = 1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + 1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix}
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Invariance

Example: Suppose $D = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $D' = \begin{pmatrix} 2 & 0 \\ 1 & 1 \end{pmatrix}$.

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\begin{pmatrix} 1 \\ 0 \end{pmatrix} = -1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + 1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} -1 \\ 1 \end{pmatrix}
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Example: Suppose \( D = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \) and \( D' = \begin{pmatrix} 2 & 0 \\ 1 & 1 \end{pmatrix} \).

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\]

\[
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\]

\[
D' = \begin{pmatrix} 2 & 0 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \end{pmatrix} = DK
\]
Invariance

Theorem: Suppose $q^*$ is an equilibrium price of $\mathcal{F}(D)$. At that price, let $\hat{z}^a$ be agent $a$'s equilibrium portfolio, which achieves consumption of $\hat{x}^a$.

Then the economy $\mathcal{F}(D')$, where $D' = DK$ for some invertible $K$, has the equilibrium price.
Invariance

Theorem: Suppose $q^*$ is an equilibrium price of $\mathcal{F}(D)$. At that price, let $\hat{z}^a$ be agent $a$’s equilibrium portfolio, which achieves consumption of $\hat{x}^a$.

Then the economy $\mathcal{F}(D')$, where $D' = DK$ for some invertible $K$, has the equilibrium price $q^*K$. 

Invariance

Theorem: Suppose \( q^* \) is an equilibrium price of \( F(D) \). At that price, let \( \hat{z}^a \) be agent \( a \)’s equilibrium portfolio, which achieves consumption of \( \hat{x}^a \).

Then the economy \( F(D') \), where \( D' = DK \) for some invertible \( K \), has the equilibrium price \( q^* K \). The equilibrium portfolio of agent \( a \) is \( K^{-1} \hat{z}^a \) and his consumption is \( \hat{x}^a \).
Invariance

**Theorem:** Suppose $q^{*}$ is an equilibrium price of $\mathcal{F}(D)$. At that price, let $\hat{z}^a$ be agent $a$’s equilibrium portfolio, which achieves consumption of $\hat{x}^a$.

Then the economy $\mathcal{F}(D')$, where $D' = DK$ for some invertible $K$, has the equilibrium price $q^{*}K$. The equilibrium portfolio of agent $a$ is $K^{-1}\hat{z}^a$ and his consumption is $\hat{x}^a$.

**Proof:** Note that $\sum_{a \in A} \hat{z}^a = 0$ implies $\sum_{a \in A} K^{-1}\hat{z}^a = 0$. 
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**Theorem:** Suppose $q^*$ is an equilibrium price of $\mathcal{F}(D)$. At that price, let $\hat{z}^a$ be agent $a$’s equilibrium portfolio, which achieves consumption of $\hat{x}^a$.

Then the economy $\mathcal{F}(D')$, where $D' = DK$ for some invertible $K$, has the equilibrium price $q^* K$. The equilibrium portfolio of agent $a$ is $K^{-1} \hat{z}^a$ and his consumption is $\hat{x}^a$.

**Proof:** Note that $\sum_{a \in A} \hat{z}^a = 0$ implies $\sum_{a \in A} K^{-1} \hat{z}^a = 0$. Furthermore, if $x \in B(q^*, \omega^a, D)$ and achieved by $z$, then $x \in B(q^* K, \omega^a, D')$ and achieved by $K^{-1} z$.

And vice versa. So $B(q^*, \omega^a, D) = B(q^* K, \omega^a, D')$. 
Invariance

Theorem: Suppose \( q^* \) is an equilibrium price of \( F(D) \). At that price, let \( \hat{z}^a \) be agent \( a \)'s equilibrium portfolio, which achieves consumption of \( \hat{x}^a \).

Then the economy \( F(D') \), where \( D' = DK \) for some invertible \( K \), has the equilibrium price \( q^* K \). The equilibrium portfolio of agent \( a \) is \( K^{-1} \hat{z}^a \) and his consumption is \( \hat{x}^a \).

Proof: Note that \( \sum_{a \in A} \hat{z}^a = 0 \) implies \( \sum_{a \in A} K^{-1} \hat{z}^a = 0 \). Furthermore, if \( x \in B(q^*, \omega^a, D) \) and achieved by \( z \), then \( x \in B(q^* K, \omega^a, D') \) and achieved by \( K^{-1} z \).

And vice versa. So \( B(q^*, \omega^a, D) = B(q^* K, \omega^a, D') \).

It follows that \( \hat{x}^a \in \arg\max_{x \in B(q^* K, \omega^a, D')} U^a(x) \) and is achieved by \( K^{-1} \hat{z}^a \).
Invariance

To show: if \( x \in B(q^*, \omega^a, D) \) and achieved by \( z \), then \( x \in B(q^* K, \omega^a, D') \) and achieved by \( K^{-1} z \).

Since \( x = (x_0, x_{-0}) \) is in \( B(q^*, \omega^a, D) \) and achieved by \( z \),

\[
\begin{align*}
x_0 & \leq \omega_0^a - q^* \cdot z \\
x_{-0} & \leq \omega_{-0}^a + D z
\end{align*}
\]
Invariance

To show: if \( x \in B(q^*, \omega^a, D) \) and achieved by \( z \), then \( x \in B(q^* K, \omega^a, D') \) and achieved by \( K^{-1} z \).

Since \( x = (x_0, x_{-0}) \) is in \( B(q^*, \omega^a, D) \) and achieved by \( z \),

\[
\begin{align*}
x_0 & \leq \omega_0^a - q^* \cdot z \\
x_{-0} & \leq \omega_{-0}^a + Dz
\end{align*}
\]

Given that \( D' = DK \), we have

\[
\begin{align*}
x_0 & \leq \omega_0^a - q^* K \cdot (K^{-1} z) \\
x_{-0} & \leq \omega_{-0}^a + D' K^{-1} z
\end{align*}
\]

\[\text{QED}\]
Arbitrage

Security prices $q \in R^S$ admits arbitrage if there is $\bar{z} \in R^S$ such that $q \cdot \bar{z} \leq 0$ and $D\bar{z} \geq 0$, with either inequality strict.
Arbitrage

Security prices \( q \in R^S \) admits arbitrage if there is \( z \in R^S \) such that 
\[ q \cdot z \leq 0 \text{ and } Dz \geq 0, \] 
with either inequality strict.

An equilibrium price cannot admit arbitrage because if \( q^* \) admits arbitrage the problem 
\[ \max_{x \in B(q^*, \omega^a, D)} U^a(x) \] 
has no solution:
Arbitrage

Security prices \( q \in R^S \) admits arbitrage if there is \( \bar{z} \in R^S \) such that \( q \cdot \bar{z} \leq 0 \) and \( D\bar{z} \geq 0 \), with either inequality strict.

An equilibrium price cannot admit arbitrage because if \( q^* \) admits arbitrage the problem \( \max_{x \in B(q^*, \omega, D)} U^a(x) \) has no solution:

for any \( x = (x_0, x_{-0}) \) in \( B(q^*, \omega, D) \), the bundle \( x + (-q \cdot \bar{z}, D\bar{z}) \) is also in \( B(q^*, \omega, D) \) and, by (P2), has a higher utility.
Arbitrage

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An equilibrium price cannot admit arbitrage because if $q^*$ admits arbitrage the problem $\max_{x \in B(q^*,\omega, D)} U^a(x)$ has no solution:

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Fundamental Theorem: If $q^* \in R^S$ admits no arbitrage, then there is $p = (p_1, p_2, ..., p_L) \gg 0$ (state prices) such that

$$q_s = p_1d_{1s} + p_2d_{2s} + ... + p_Ld_{Ls}.$$
Arbitrage

Security prices $q \in R^S$ admits arbitrage if there is $\bar{z} \in R^S$ such that $q \cdot \bar{z} \leq 0$ and $D\bar{z} \geq 0$, with either inequality strict.

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**Fundamental Theorem:** If $q^* \in R^S$ admits no arbitrage, then there is $p = (p_1, p_2, ..., p_L) \gg 0$ (state prices) such that

$$q_s = p_1d_{1s} + p_2d_{2s} + ... + p_Ld_{Ls}.$$ 

More succinctly, $q = pD$. 

Lectures on General Equilibrium Theory ⋆ ⋆ ⋆ Equilibrium in a financial economy – p. 19/25
Characterizing financial equilibria

The Fundamental Theorem gives us a simple way of understanding the budget set of the agent. Assuming that utility is strongly monotone, agent chooses contingent consumption from the set

\[
L(q, \omega) = \left\{ x \in R^{L+1}_+ : \begin{array}{l}
x_0 = \omega_0 - q \cdot z \\
x_{-0} = \omega_{-0} + Dz \\
\text{for some } z \in R^S \end{array} \right\}.
\]

Proposition: Suppose that \( q \in R^S \) is a no arbitrage price and \( q = pD \) for \( p \in R^{L+}_+ \). Then \( L(q, \omega) = \tilde{L}(p, \omega) \) where

\[
\tilde{L}(p, \omega) \equiv \{ x \in R^{L+1}_+ : x_0 + p_{-0} \cdot x_{-0} = \omega_0 + p_{-0} \cdot \omega_{-1} \} \cap \{ x \in R^{L+1}_+ : x_{-0} = \omega_{-0} + Dz \text{ for some } z \in R^S \}.
\]

In other words, \( L(q, \omega) \) is the classic (complete market) budget plane at price \( (1, p) \in R^{L+1}_{++} \), restricted to the space spanned by the assets.
Characterizing financial equilibria

Define \( \tilde{x}^a(p) = \arg\max_{x \in L(p,\omega^a)} U^a(x) \).

A price \( p^* \) is a no-arbitrage equilibrium price of this economy if

\[
\sum_{a \in A} \tilde{x}^a(p^*) = \bar{\omega}.
\]
Characterizing financial equilibria

Define \( \tilde{x}^a(p) = \arg\max_{x \in \tilde{L}(p,\omega^a)} U^a(x) \).

A price \( p^* \) is a no-arbitrage equilibrium price of this economy if

\[
\sum_{a \in A} \tilde{x}^a(p^*) = \bar{\omega}.
\]

**Proposition:** (i) If \( q^* \) is a financial equilibrium price and \( q^* = p^* D \) for \( p^* \gg 0 \), then \( p^* \) is a no-arbitrage equilibrium price.

(ii) If \( p^* \) is no-arbitrage equilibrium price, then \( q^* = p^* D \) is an equilibrium price of the financial economy.
Characterizing financial equilibria

Define \( \tilde{x}^a(p) = \arg\max_{x \in \tilde{L}(p, \omega^a)} U^a(x) \).

A price \( p^* \) is a no-arbitrage equilibrium price of this economy if

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\sum_{a \in A} \tilde{x}^a(p^*) = \bar{\omega}.
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Proposition: (i) If \( q^* \) is a financial equilibrium price and \( q^* = p^* D \) for \( p^* \gg 0 \), then \( p^* \) is a no-arbitrage equilibrium price.

(ii) If \( p^* \) is no-arbitrage equilibrium price, then \( q^* = p^* D \) is an equilibrium price of the financial economy.

Proofs of equilibrium existence first prove the existence of a no-arbitrage equilibrium and then appeal to (ii) to obtain a financial equilibrium.
Characterizing financial equilibria

Define $\tilde{x}^a(p) = \arg\max_{x \in \tilde{L}(p, \omega^a)} U^a(x)$. A price $p^*$ is a no-arbitrage equilibrium price of this economy if $\sum_{a \in A} \tilde{x}^a(p^*) = \bar{\omega}$.

Proposition: (i) If $q^*$ is a financial equilibrium price and $q^* = p^*D$ for $p^* \gg 0$, then $p^*$ is a no-arbitrage equilibrium price.

Proof of (i): Since $\tilde{L}(p, \omega^a) = L(q, \omega^a)$ when $q = pD$,

$$\tilde{x}^a(p) = \hat{x}^a(q) \text{ where } \hat{x}^a(q) = \arg\max_{x \in L(q, \omega^a)} U^a(x),$$

i.e., $\hat{x}^a(q)$ is agent $a$’s demand for contingent consumption at $q$ (as defined previously). In particular, $\tilde{x}^a(p^*) = \hat{x}^a(q^*)$. We have already shown that the ‘real’ economy clears at a financial equilibrium, i.e.,

$$\sum_{a \in A} \hat{x}^a(q^*) = \bar{\omega},$$

so we also have $\sum_{a \in A} \tilde{x}^a(p^*) = \bar{\omega}$. QED
Characterizing financial equilibria

Proposition: (ii) If $p^*$ is no arbitrage equilibrium price, then $q^* = p^* D$ is an equilibrium price of the financial economy.

Proof: Consider all agents but one, $\bar{a}$. For all agents other than $\bar{a}$, there is $z^a$ such that $x^a_0(p^*) = \omega^a_0 + Dz^a$. So $z^a$ is agent $a$’s demand for securities at $q^* = p^* D$. 
Characterizing financial equilibria

Proposition: (ii) If $p^*$ is no arbitrage equilibrium price, then $q^* = p^* D$ is an equilibrium price of the financial economy.

Proof: Consider all agents but one, $\bar{a}$. For all agents other than $\bar{a}$, there is $z^a$ such that $x^a_0(p^*) = \omega^a_0 + Dz^a$. So $z^a$ is agent $a$’s demand for securities at $q^* = p^* D$. Summing across agents (except $\bar{a}$), we obtain

$$\sum_{a \in A \setminus \{\bar{a}\}} [\hat{x}^a_0(p^*) - \omega^a_0] = D \left( \sum_{a \in A \setminus \bar{a}} z^a \right).$$

Since $p^*$ is a no-arbitrage equilibrium price,

$$\sum_{a \in A \setminus \{\bar{a}\}} [\hat{x}^a_0(p^*) - \omega^a_0] = \omega^\bar{a}_0 - \hat{x}^\bar{a}_0(p^*).$$
Characterizing financial equilibria

Proposition: (ii) If \( p^\ast \) is no arbitrage equilibrium price, then \( q^\ast = p^\ast D \) is an equilibrium price of the financial economy.

Proof: Consider all agents but one, \( \bar{a} \). For all agents other than \( \bar{a} \), there is \( z^a \) such that \( x_{-0}^a(p^\ast) = \omega_{-0}^a + D z^a \). So \( z^a \) is agent \( a \)'s demand for securities at \( q^\ast = p^\ast D \). Summing across agents (except \( \bar{a} \)), we obtain

\[
\sum_{a \in A \setminus \{\bar{a}\}} \left[ \hat{x}_{-0}^a(p^\ast) - \omega_{-0}^a \right] = D \left( \sum_{a \in A \setminus \bar{a}} z^a \right).
\]

Since \( p^\ast \) is a no-arbitrage equilibrium price,

\[
\sum_{a \in A \setminus \{\bar{a}\}} \left[ \hat{x}_{-0}^a(p^\ast) - \omega_{-0}^a \right] = \omega_{-0}^\bar{a} - \hat{x}_{-0}^\bar{a}(p^\ast).
\]

Therefore, \( \hat{x}_{-0}^\bar{a}(p^\ast) = \omega_{-0}^\bar{a} + D \left( -\sum_{a \in A \setminus \{\bar{a}\}} z^a \right) \).
Characterizing financial equilibria

Proposition: (ii) If \( p^* \) is no arbitrage equilibrium price, then \( q^* = p^* D \) is an equilibrium price of the financial economy.

Proof: Consider all agents but one, \( \bar{a} \). For all agents other than \( \bar{a} \), there is \( z^a \) such that \( x^a_0(p^*) = \omega^a_0 + Dz^a \). So \( z^a \) is agent \( a \)'s demand for securities at \( q^* = p^* D \). Summing across agents (except \( \bar{a} \)), we obtain

\[
\sum_{a \in A \setminus \{\bar{a}\}} \left[ \hat{x}^a_0(p^*) - \omega^a_0 \right] = D \left( \sum_{a \in A \setminus \bar{a}} z^a \right).
\]

Since \( p^* \) is a no-arbitrage equilibrium price,

\[
\sum_{a \in A \setminus \{\bar{a}\}} \left[ \hat{x}^a_0(p^*) - \omega^a_0 \right] = \omega_{-0}^\bar{a} - \hat{x}_{-0}^\bar{a}(p^*).
\]

Therefore, \( \hat{x}_{-0}^\bar{a}(p^*) = \omega_{-0}^\bar{a} + D \left( - \sum_{a \in A \setminus \{\bar{a}\}} z^a \right) \). In other words, the demand for securities of agent \( \bar{a} \) is \( z^\bar{a} = - \sum_{a \in A \setminus \{\bar{a}\}} z^a \) and we conclude that \( q^* \) is a financial equilibrium price.

QED
The existence of financial equilibria

Theorem: Suppose agents in a financial economy have utility functions obeying (P1), (P2), and (P3) and that there is some agent $\bar{a}$ with endowment $\omega^{\bar{a}} \gg 0$. Then this economy has a financial equilibrium.

Proof sketch: We prove this by showing that a no-arbitrage equilibrium exists.

For agent $\bar{a}$, let $\bar{x}^{\bar{a}}(p)$ maximize $U^{\bar{a}}(x)$ subject to

$$x \in \{ x \in \mathbb{R}_+^{L+1} : x_0 + p \cdot x_{-0} = \omega^0_{\bar{a}} + p \cdot \omega_{\bar{a}_1} \}.$$ 

In other words, this is agent $\bar{a}$’s demand if the market is complete.

Define $Z : \mathbb{R}_+^L \rightarrow \mathbb{R}_+^{L+1}$ by

$$\bar{Z}(p) = \sum_{a \in A \setminus \{\bar{a}\}} \hat{x}^{a}(p) + \bar{x}^{\bar{a}}(p) - \bar{\omega}.$$ 

For $a \in A \setminus \{\bar{a}\}$, $\hat{x}^{a}$ is a continuous function that obeys the budget identity but it need not satisfy the boundary condition.

On the other hand, $\bar{x}^{\bar{a}}$ is continuous, obeys the budget identity, and satisfies the boundary condition because its budget set is classical and $\omega^{\bar{a}} \gg 0$ (so $\bar{a}$’s income never approaches zero).
The existence of financial equilibria

Therefore, the function

\[ \bar{Z}(p) = \sum_{a \in A \setminus \{\bar{a}\}} \hat{x}^a(p) + \bar{x}^\bar{a}(p) - \bar{w}. \]

is continuous and bounded below and obeys Walras’ Law and the boundary condition. Thus there is \( p^* \) such that \( \bar{Z}(p^*) = 0 \)

I claim that \( \bar{x}^\bar{a}(p^*) = \hat{x}^\bar{a}(p^*) \); this is sufficient to guarantee that \( p^* \) is a no-arbitrage equilibrium price. I simply need to show that

\[ \bar{x}^\bar{a}_0(p^*) = \omega^\bar{a}_0 + Dz \text{ for some } z \in R^S. \]

This is true since, for all \( a \in A \setminus \{\bar{a}\}, \)

\[ \hat{x}^a_0(p^*) - \omega^a_0 = Dz^a \text{ for some } z^a \in R^S \]

and it follows that

\[ \bar{x}^\bar{a}_0(p^*) - \omega^\bar{a}_0 = - \sum_{a \in A \setminus \{\bar{a}\}} \left[ \hat{x}^a_0(p^*) - \omega^a_0 \right] \]

\[ = D \left( - \sum_{a \in A \setminus \{\bar{a}\}} z^a \right) \]

since \( \bar{Z}_0(p^*) = 0. \)