Lectures on

General Equilibrium Theory

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Equilibrium in a financial economy

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Financial Assets

Assume that there are two dates, Today (date 0) and Tomorrow (date 1). There are L states of the world tomorrow.

Security/Asset is a promise of payment (positive or negative), conditional on the realization of the state.

We write payoff of security s as a column vector, called the payoff vector of security s:

$$\begin{pmatrix} d_{1s} \\ d_{2s} \\ d_{3s} \\ . \\ . \\ d_{Ls} \end{pmatrix}$$

Payoff Matrix

If economy has S securities (called 1, 2,..., S), then the payoff matrix is

$$D = \begin{pmatrix} d_{11} & d_{12} & . & . & d_{1S} \\ d_{21} & d_{22} & . & . & d_{2S} \\ d_{31} & d_{32} & . & . & d_{3S} \\ . & . & . & . & . \\ . & . & . & . & . \\ d_{L1} & d_{L2} & . & . & d_{LS} \end{pmatrix}.$$

Trade in securities/portfolio represented by the column vector z in R^S.

If $z_s > 0$ then agent is buying asset s;
if $z_s < 0$ then agent is selling asset s.

Contingent consumption

A portfolio z (in R^S) gives a payoff in state i of

$$z_1 d_{i1} + z_2 d_{i2} + ... z_S d_{iS}.$$

This is the ith entry in the column vector Dz in R^L since

$$Dz = z_1 \begin{pmatrix} d_{11} \\ d_{21} \\ . \\ . \\ d_{L1} \end{pmatrix} + z_2 \begin{pmatrix} d_{12} \\ d_{22} \\ . \\ . \\ d_{L2} \end{pmatrix} + ... z_S \begin{pmatrix} d_{1S} \\ d_{2S} \\ . \\ . \\ d_{LS} \end{pmatrix}.$$

So Dz is the payoff vector of the portfolio z.

The subspace $\{Dz; z \in R^S\}$, denoted by Span(D), is called the span of the security payoffs or the asset span.

Incomplete Markets

If Span(D) is a *strict* subspace of R^L, then the economy has incomplete markets.

Suppose agent has endowment $\omega = (\omega_0, \omega_1, ..., \omega_L) \in R_+^{L+1}$,

where ω_0 in R_+ is agent's endowment at $t = 0$ and $\omega_{-0} = (\omega_1, \omega_2, ..., \omega_L)$ is his endowment at $t = 1$.

With a portfolio $\bar{z}$, the agent's contingent consumption at $t = 1$ is $\omega_{-0} + D\bar{z}$.

With incomplete markets, there are contingent consumption bundles that the agent cannot achieve even if he could assemble any portfolio he likes, i.e.,

there is $x^* \in R_+^L$ such that there is no z with $\omega_{-0} + Dz = x^*$.

Incomplete Markets

Example: Suppose there are three states of the world at $t = 1$ and just two securities, 1 and 2, with payoff vectors $(1, 1, 0)$ and $(0, 1, 2)$ respectively.

An agent's endowment at $t = 1$ is $\omega_{-0} = (2, 3, 0)$. The portfolio $z = (z_1, z_2)$ gives this agent contingent consumption of

$$\omega_{-0} + Dz = \begin{pmatrix} 2 \\ 3 \\ 0 \end{pmatrix} + z_1 \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + z_2 \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}.$$

Note that $M = \{\omega_{-0} + Dz : z \in R^2\}$ is a plane in R^3 passing through the point ω_{-0}.

In particular $R^3_+ \not\subseteq M$ – so market is incomplete.

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$$U(x_0, x_{-0}) = u(x_0) + \delta\left[\pi_1 v(x_1) + \pi_2 v(x_2) + ... + \pi_L v(x_L)\right].$$

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Suppose $q \in R^s$ is the asset price vector.

Then the agent's budget set is

$$B(q, \omega, D) = \left\{\begin{array}{ll} x \in R_+^{L+1} : & x_0 \leq \omega_0 - q \cdot z \\ & x_{-0} \leq \omega_{-0} + Dz \\ & \text{for some } z \in R^S \end{array}\right\}.$$

Note that the price of Today's consumption is normalized at Today 1.

Agent's utility maximization problem

Agent maximizes $U(x)$ subject to x in $B(q, \omega, D)$.

Then $\hat{x} \in \text{argmax}_{x \in B(q,\omega,D)} U(x)$ is the agent's demand for contingent consumption and $\hat{z}$ such that

$$\begin{aligned} \hat{x}_0 &\leq \omega_0 - q \cdot \hat{z} \text{ and} \\ \hat{x}_{-0} &\leq \omega_{-0} + D\hat{z} \end{aligned}$$

is the agent's demand for securities.

Note: if U is strictly monotone (obeys (P2)), then the inequalities above are equalities.

Example

Agent's utility is $U(x_0, x_1, x_2) = \ln x_0 + \frac{1}{2} \ln x_1 + \frac{1}{2} \ln x_2$

Two securities with payoff vectors $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ respectively.

Price is $q = (1, 2)$. Endowment $\omega = (3, 0, 0)$.

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$$\text{Budget Set} = \left\{ \begin{array}{ll} x \in R^3_+ : & x_0 \leq 3 - z_1 - 2z_2 \\ & x_1 \leq z_1 + z_2 \\ & x_2 \leq z_2 \ \text{ for some } z = (z_1, z_2) \end{array} \right\}.$$

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Solution: $\bar{z}_1 = 0$, $\bar{z}_2 = \frac{3}{4}$, $\bar{x}_0 = \frac{3}{2}$, $\bar{x}_1 = \frac{3}{4}$, $\bar{x}_2 = \frac{3}{4}$.

The Financial Economy

A financial economy $\mathcal{F}$ consists of a payoff matrix D and a set A of agents, each of whom has an endowment $\omega^a = (\omega_0^a, \omega_{-0}^a)$ in R_+^{L+1} and a utility function $U^a : R_+^{L+1} \to R$.

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(i) for each agent a, there is $\hat{z}^a$ such that $D\hat{z}^a$ maximizes $U^a(x)$ in $B(q^*, \omega^a, D)$ and

(ii) $\sum_{a \in A} \hat{z}^a = 0$.

Equilibrium in a financial economy

Proposition: *Suppose that q^* is an equilibrium price of $\mathcal{F}$ and let $\hat{z}^a$ and $\hat{x}^a$ be agent a's asset and consumption demand respectively. Then, provided U^a obeys (P2),*

$$\sum_{a\in A} \hat{x}^a = \sum_{a\in A} \omega^a.$$

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Proof: For agent a, we have $\hat{x}_0^a = \omega_0 - q^* \cdot \hat{z}^a$. Sum across a, we obtain

$$\sum_{a \in A} \hat{x}_0^a = \sum_{a \in A} \omega_0^a.$$

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$$\sum_{a \in A} \hat{x}_0^a = \sum_{a \in A} \omega_0^a.$$

Since $\hat{x}^a{}_{-0} = \hat{\omega}^a{}_{-0} + D\hat{z}^a$, summing across a gives us

$$\sum_{a \in A} \hat{x}^a{}_{-0} = \sum_{a \in A} \omega^a{}_{-0}.$$

. **QED**

Constrained Pareto optimality

The allocation $\{x^a\}_{a \in A}$ is feasible if $\sum_{a \in A} x^a = \sum_{a \in A} \omega^a$.

It is a constrained feasible allocation if it is feasible and there is $\{z^a\}_{a \in A}$ such that $\sum_{a \in A} z^a = 0$ and

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An allocation $\{x^a\}_{a\in A}$ is constrained Pareto optimal if there does not exist a constrained feasible allocation $\{\bar{x}^a\}_{a\in A}$ that is Pareto superior, i.e.,

$$U^a(\bar{x}^a) \geq U^a(x^a)$$

for all a and the inequality is strict for at least one agent.

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Theorem [First welfare theorem for financial economies]:
Suppose that U^a obeys (P2) for all a. Then every equilibrium allocation is constrained Pareto optimal.

Constrained Pareto optimality

Proof: Let q^* be the equilibrium price and $\{\hat{x}^a\}_{a \in A}$ the equilibrium allocation. Assume that $\{\bar{x}^a\}_{a \in A}$ is a Pareto superior constrained feasible allocation.

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For some agent $\tilde{a}$, we have $U^{\tilde{a}}(\bar{x}^{\tilde{a}}) > U^{\tilde{a}}(\hat{x}^{\tilde{a}})$.
So the bundle $\bar{x}^{\tilde{a}}$ cannot be in $\tilde{a}$'s budget set - if it were, agent $\tilde{a}$ would have chosen this bundle instead of $\hat{x}^{\tilde{a}}$.

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$$\bar{x}_0^{\tilde{a}} > \omega_0^{\tilde{a}} - q^* \cdot \bar{z}^{\tilde{a}}.$$

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Suppose not, then there is $\epsilon > 0$ such that $\bar{x}_0^a + \epsilon < \omega_0^a - q^* \cdot \bar{z}^a$.
So $\tilde{x}^a = \bar{x}^a + (\epsilon, 0, ..., 0)$ is in $B(q^*, \omega, D)$ and

$$U^a(\tilde{x}^a) > U^a(\bar{x}^a) \geq U^a(\hat{x}^a).$$

This cannot happen since $\hat{x}^a$ is a's demand.

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Summing across a, we obtain

$$\begin{aligned} \sum_{a \in A} \bar{x}_0^a &> \sum_{a \in A} \omega_0^a - q^* \cdot \left(\sum_{a \in A} \bar{z}^a \right) \\ &= \sum_{a \in A} \omega_0^a, \end{aligned}$$

which is a contradiction. **QED**

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Lemma: Let $\{1, 2, ..., S\}$ be a set of securities with no redundant securities and let D be its payoff matrix. Let there be another set of $|S|$ securities, with payoff matrix D' such that Span (D)=Span (D').

Then there is invertible matrix $S \times S$ matrix K such that $DK = D'$.

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Proof: Exercise!

Invariance

Example: Suppose $D = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $D' = \begin{pmatrix} 2 & 0 \\ 1 & 1 \end{pmatrix}$.

$$\begin{pmatrix} 2 \\ 1 \end{pmatrix} = 1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + 1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

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$$\begin{aligned} D' = \begin{pmatrix} 2 & 0 \\ 1 & 1 \end{pmatrix} &= \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \\ &= DK \end{aligned}$$

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Theorem: Suppose q^* is an equilibrium price of $\mathcal{F}(D)$. At that price, let $\hat{z}^a$ be agent a's equilibrium portfolio, which achieves consumption of $\hat{x}^a$.

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if $x \in B(q^*, \omega^a, D)$ and achieved by z, then $x \in B(q^*K, \omega^a, D')$ and achieved by $K^{-1}z$.

And vice versa. So $B(q^*, \omega^a, D) = B(q^*K, \omega^a, D')$.

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And vice versa. So $B(q^*, \omega^a, D) = B(q^*K, \omega^a, D')$.
It follows that $\hat{x}^a \in \text{argmax}_{x\in B(q^*K,\omega^a,D')}U^a(x)$ and is achieved by $K^{-1}\hat{z}^a$.

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To show: if $x \in B(q^*, \omega^a, D)$ and achieved by z, then $x \in B(q^* K, \omega^a, D')$ and achieved by $K^{-1}z$.

Since $x = (x_0, x_{-0})$ is in $B(q^*, \omega^a, D)$ and achieved by z,

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Given that $D' = DK$, we have

$$\begin{aligned} x_0 &\leq \omega_0^a - q^* K \cdot (K^{-1} z) \\ x_{-0} &\leq \omega_{-0}^a + D' K^{-1} z \end{aligned}$$

. **QED**

Arbitrage

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for any $x = (x_0, x_{-0})$ in $B(q^*, \omega, D)$, the bundle $x + (-q \cdot \bar{z}, D\bar{z})$ is also in $B(q^*, \omega, D)$ and, by (P2), has a higher utility.

Arbitrage

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Fundamental Theorem: If $q^* \in R^S$ admits no arbitrage, then there is $p = (p_1, p_2, ..., p_L) \gg 0$ (state prices) such that

$$q_s = p_1 d_{1s} + p_2 d_{2s} + ... + p_L d_{Ls}.$$

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More succinctly, $q = pD$.

Characterizing financial equilibria

The Fundamental Theorem gives us a simple way of understanding the budget set of the agent. Assuming that utility is strongly monotone, agent chooses contingent consumption from the set

$$L(q,\omega) = \left\{ \begin{array}{ll} x \in R_+^{L+1} : & x_0 = \omega_0 - q \cdot z \\ & x_{-0} = \omega_{-0} + Dz \\ & \text{for some } z \in R^S \end{array} \right\}.$$

Proposition: Suppose that $q \in R^S$ is a no arbitrage price and $q = pD$ for $p \in R^L_{++}$. Then $L(q,\omega) = \tilde{L}(p,\omega)$ where

$\tilde{L}(p,\omega) \equiv \{x \in R_+^{L+1} : x_0 + p_{-0} \cdot x_{-0} = \omega_0 + p_{-0} \cdot \omega_{-1})\} \cap$
$\{x \in R_+^{L+1} : x_{-0} = \omega_{-0} + Dz$ for some $z \in R^S\}$.

In other words, $L(q,\omega)$ is the classic (complete market) budget plane at price $(1,p) \in R_{++}^{L+1}$, restricted to the space spanned by the assets.

Characterizing financial equilibria

Define $\tilde{x}^a(p) = \text{argmax}_{x \in \tilde{L}(p,\omega^a)} U^a(x)$.

A price p^* is a no-arbitrage equilibrium price of this economy if

$$\sum_{a \in A} \tilde{x}^a(p^*) = \bar{\omega}.$$

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Proposition: (i) If q^* is a financial equilibrium price and $q^* = p^* D$ for $p^* \gg 0$, then p^* is a no-arbitrage equilibrium price.
(ii) If p^* is no-arbitrage equilibrium price, then $q^* = p^* D$ is an equilibrium price of the financial economy.

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Proofs of equilibrium existence first prove the existence of a no-arbitrage equilibrium and then appeal to (ii) to obtain a financial equilibrium.

Characterizing financial equilibria

Define $\tilde{x}^a(p) = \text{argmax}_{x \in \tilde{L}(p,\omega^a)} U^a(x)$. A price p^* is a no-arbitrage equilibrium price of this economy if $\sum_{a \in A} \tilde{x}^a(p^*) = \bar{\omega}$.

Proposition: (i) If q^* is a financial equilibrium price and $q^* = p^* D$ for $p^* \gg 0$, then p^* is a no-arbitrage equilibrium price.

Proof of (i): Since $\tilde{L}(p, \omega^a) = L(q, \omega^a)$ when $q = pD$,

$$\tilde{x}^a(p) = \hat{x}^a(q) \text{ where } \hat{x}^a(q) = \text{argmax}_{x \in L(q,\omega^a)} U^a(x),$$

i.e., $\hat{x}^a(q)$ is agent a's demand for contingent consumption at q (as defined previously). In particular, $\tilde{x}^a(p^*) = \hat{x}^a(q^*)$. We have already shown that the 'real' economy clears at a financial equilibrium, i.e.,

$$\sum_{a \in A} \hat{x}^a(q^*) = \bar{\omega},$$

so we also have $\sum_{a \in A} \tilde{x}^a(p^*) = \bar{\omega}$. **QED**

Characterizing financial equilibria

Proposition: (ii) If p^* is no arbitrage equilibrium price, then $q^* = p^* D$ is an equilibrium price of the financial economy.

Proof: Consider all agents but one, $\bar{a}$. For all agents other than $\bar{a}$, there is z^a such that $x^a_{-0}(p^*) = \omega^a_{-0} + Dz^a$. So z^a is agent a's demand for securities at $q^* = p^* D$.

Characterizing financial equilibria

Proposition: (ii) If p^* is no arbitrage equilibrium price, then $q^* = p^* D$ is an equilibrium price of the financial economy.

Proof: Consider all agents but one, $\bar{a}$. For all agents other than $\bar{a}$, there is z^a such that $x^a_{-0}(p^*) = \omega^a_{-0} + Dz^a$. So z^a is agent a's demand for securities at $q^* = p^* D$. Summing across agents (except $\bar{a}$), we obtain

$$\sum_{a \in A \setminus \{\bar{a}\}} \left[\hat{x}^a_{-0}(p^*) - \omega^a_{-0}\right] = D \left(\sum_{a \in A \setminus \bar{a}} z^a \right).$$

Since p^* is a no-arbitrage equilibrium price,

$$\sum_{a \in A \setminus \{\bar{a}\}} \left[\hat{x}^a_{-0}(p^*) - \omega^a_{-0}\right] = \omega^{\bar{a}}_{-0} - \hat{x}^{\bar{a}}_{-0}(p^*).$$

Characterizing financial equilibria

Proposition: (ii) If p^* is no arbitrage equilibrium price, then $q^* = p^*D$ is an equilibrium price of the financial economy.

Proof: Consider all agents but one, $\bar{a}$. For all agents other than $\bar{a}$, there is z^a such that $x^a_{-0}(p^*) = \omega^a_{-0} + Dz^a$. So z^a is agent a's demand for securities at $q^* = p^*D$. Summing across agents (except $\bar{a}$), we obtain

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Since p^* is a no-arbitrage equilibrium price,

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Therefore, $\hat{x}^{\bar{a}}_{-0}(p^*) = \omega^{\bar{a}}_{-0} + D\left(-\sum_{a \in A \setminus \{\bar{a}\}} z^a\right).$

Characterizing financial equilibria

Proposition: (ii) If p^* is no arbitrage equilibrium price, then $q^* = p^*D$ is an equilibrium price of the financial economy.

Proof: Consider all agents but one, $\bar{a}$. For all agents other than $\bar{a}$, there is z^a such that $x^a_{-0}(p^*) = \omega^a_{-0} + Dz^a$. So z^a is agent a's demand for securities at $q^* = p^*D$. Summing across agents (except $\bar{a}$), we obtain

$$\sum_{a \in A \setminus \{\bar{a}\}} \left[\hat{x}^a_{-0}(p^*) - \omega^a_{-0}\right] = D \left(\sum_{a \in A \setminus \bar{a}} z^a \right).$$

Since p^* is a no-arbitrage equilibrium price,

$$\sum_{a \in A \setminus \{\bar{a}\}} \left[\hat{x}^a_{-0}(p^*) - \omega^a_{-0}\right] = \omega^{\bar{a}}_{-0} - \hat{x}^{\bar{a}}_{-0}(p^*).$$

Therefore, $\hat{x}^{\bar{a}}_{-0}(p^*) = \omega^{\bar{a}}_{-0} + D\left(-\sum_{a \in A \setminus \{\bar{a}\}} z^a\right)$. In other words, the demand for securities of agent $\bar{a}$ is $z^{\bar{a}} = -\sum_{a \in A \setminus \{\bar{a}\}} z^a$ and we conclude that q^* is a financial equilibrium price. **QED**

The existence of financial equilibria

Theorem: Suppose agents in a financial economy have utility functions obeying (P1), (P2), and (P3) and that there is some agent $\bar{a}$ with endowment $\omega^{\bar{a}} \gg 0$. Then this economy has a financial equilibrium.

Proof sketch: We prove this by showing that a no-arbitrage equilibrium exists.

For agent $\bar{a}$, let $\bar{x}^{\tilde{a}}(p)$ maximize $U^{\bar{a}}(x)$ subject to

$$x \in \{x \in R_{+}^{L+1} : x_0 + p \cdot x_{-0} = \omega_0^{\bar{a}} + p \cdot \omega_{-1}^{\bar{a}})\}.$$

In other words, this is agent $\bar{a}$'s demand if the market is complete.
Define $Z : R_{++}^{L} \to R^{L+1}$ by

$$\bar{Z}(p) = \sum_{a \in A \setminus \{\bar{a}\}} \hat{x}^{a}(p) + \bar{x}^{\bar{a}}(p) - \bar{\omega}.$$

For $a \in A \setminus \{\bar{a}\}$, $\hat{x}^a$ is a continuous function that obeys the budget identity but it need not satisfy the boundary condition.

On the other hand, $\bar{x}^{\bar{a}}$ is continuous, obeys the budget identity, *and* satisfies the boundary condition because its budget set is classical and $\omega^{\bar{a}} \gg 0$ (so $\bar{a}$'s income never approaches zero).

The existence of financial equilibria

Therefore, the function

$$\bar{Z}(p) = \sum_{a \in A \setminus \{\bar{a}\}} \hat{x}^a(p) + \bar{x}^{\bar{a}}(p) - \bar{\omega}.$$

is continuous and bounded below and obeys Walras' Law and the boundary condition. Thus there is p^* such that $\bar{Z}(p^*) = 0$

I claim that $\bar{x}^{\bar{a}}(p^*) = \hat{x}^{\bar{a}}(p^*)$; this is sufficient to guarantee that p^* is a no-arbitrage equilibrium price. I simply need to show that

$$\bar{x}^{\bar{a}}{}_{-0}(p^*) = \omega^{\bar{a}}{}_{-0} + Dz \text{ for some } z \in R^S.$$

This is true since, for all $a \in A \setminus \{\bar{a}\}$,

$$\hat{x}^a{}_{-0}(p^*) - \omega^a{}_{-0} = Dz^a \text{ for some } z^a \in R^S$$

and it follows that

$$\begin{aligned} \bar{x}^{\bar{a}}{}_{-0}(p^*) - \omega^{\bar{a}}{}_{-0} &= - \sum_{a \in A \setminus \{\bar{a}\}} \left[\hat{x}^a{}_{-0}(p^*) - \omega^a{}_{-0}\right] \\ &= D\left(- \sum_{a \in A \setminus \{\bar{a}\}} z^a\right) \end{aligned}$$

since $\bar{Z}_{-0}(p^*) = 0$.

QED