Comparative statics with linear objectives: normal demand, monotone marginal costs and ranking multi-prior beliefs

Pawel Dziewulski and John K.-H. Quah

Lattices and Supermodularity

Definition: A partially ordered set $(X, \geqslant_X)$ is a lattice if every two elements has a least upper bound (supremum) and a greatest lower bound (infimum).

We denote the supremum of x and y by $x \vee y$ and their infimum by $x \wedge y$.

Example 1: $(\mathbb{R}^\ell, \geqslant)$ is a lattice, where $\geqslant$ is the product order, i.e. $x \geqslant y$ if $x_i \geqslant y_i$ for $i = 1, 2, .., \ell$. Indeed,

$$\begin{aligned} x \vee y &= (\max\{x_1, y_1\}, \max\{x_2, y_2\}, ..., \max\{x_\ell, y_\ell\}) \\ x \wedge y &= (\min\{x_1, y_1\}, \min\{x_2, y_2\}, ..., \min\{x_\ell, y_\ell\}). \end{aligned}$$

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Example 2: Distributions on $S \subset \mathbb{R}$ is a lattice when ordered by first order stochastic dominance.

$$\begin{aligned} (\lambda \vee \lambda')(s) &= \min\{\lambda(s), \lambda'(s)\} \\ (\lambda \wedge \lambda')(s) &= \max\{\lambda(s), \lambda'(s)\} \end{aligned}$$

Lattices and Supermodularity

Let $(X, \geqslant_X)$ be a lattice.

A function $f : (X, \geqslant_X) \to \mathbb{R}$ is a supermodular function if, for any x, x' in X,

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For $f : (\mathbb{R}^\ell_+, \geqslant) \to \mathbb{R}$, supermodularity is equivalent to the following:

for any $i \in \{1, 2, ..., \ell\}$, with $x_i'' > x_i'$,

$$f(x_i'', x_{-i}) - f(x_i', x_{-i}) \text{ is increasing in } x_{-i}.$$

If f is a production function, this says that the marginal productivity of factor i increases as the input of other factors, x_{-i} is raised.

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If f is differentiable, the supermodularity of f is equivalent to

$$\frac{\partial^2 f}{\partial x_i \partial x_j}(x) \geqslant 0 \text{ for all } x, \text{ and } i \neq j.$$

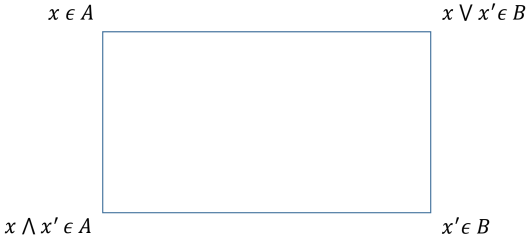
Background and Motivation

Let A and B be two nonempty subsets in lattice $(X, \geqslant_X)$.

B dominates A in the strong set order if for any $x \in A$ and $x' \in B$,

$$x \vee x' \in B \text{ and } x \wedge x' \in A.$$

We denote this by $B \geq A$.



This implies that B is higher than A in the following sense:

for any $x \in A$, there is $\hat{x} \in B$ such that $\hat{x} \geqslant x$ (choose $\hat{x} = x \vee x'$) and

for any $x' \in B$ there is $\tilde{x} \in A$ such that $x' \geqslant \tilde{x} \in B$ (choose $\tilde{x} = x \wedge x'$).

Background and Motivation

Basic MCS Theorem.

Suppose $\eta : X \to \mathbb{R}$ is a supermodular function and $B \geq A$. Then

$$\mathrm{argmax}_{x \in B}\, \eta(x) \;\succeq\; \mathrm{argmax}_{x \in A}\, \eta(x).$$

Background and Motivation

Basic MCS Theorem.

Suppose $\eta : X \to \mathbb{R}$ is a supermodular function and $B \succeq A$. Then

$$\operatorname{argmax}_{x \in B} \eta(x) \;\succeq\; \operatorname{argmax}_{x \in A} \eta(x).$$

Example. A firm maximizes profit $\eta(x) = F(x) - p \cdot x$,

where $x \in \mathbb{R}^{\ell}_{+}$ is the vector of inputs and $p \in \mathbb{R}^{\ell}_{++}$ are the factor prices.

Let $A = \{x \in \mathbb{R}^{\ell}_{+} : x_1 \leqslant M\}$ and $B = \mathbb{R}^{\ell}_{+}$. Then $B \succeq A$.

If F is supermodular, then so is η is supermodular.

Theorem says that more of *all* factors will be employed if the constraint on increasing factor 1 is removed.

Background and Motivation

Application of this result is limited because constraint sets are often not ranked according to the strong set order.

Example. At price $p \in \mathbb{R}^{\ell}_{++}$ and income $w > 0$, a consumer's budget set is $B(p, w) = \{x \in \mathbb{R}^{\ell}_{+} : p \cdot x \leqslant w\}$.
Suppose $w'' > w'$: it is *not* the case that $B(p, w'') \geq B(p, w')$.

Indeed, if $x' \in B(p, w')$ and $x'' \in B(p, w'')$, it is possible that $p \cdot (x' \vee x'') > w''$ and hence $x' \vee x'' \notin B(p, w'')$.

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Quah (2007, Ecta) deals with this issue: obtains comparative statics by weakening the notion of set comparisons and strengthening the requirements on the objective function η.

This approach leads to a sufficient, but not necessary, condition for normal demand.

This paper provides a *characterization* of normal demand.

Background and Motivation

We strengthen the condition on the objective function η even more.

We assume η is a *linear function*.

Question: what condition linking A and B guarantees that

$\mathrm{argmax}_{x \in B}\, \eta(x)$ is higher than $\mathrm{argmax}_{x \in A}\, \eta(x)$ (in some natural sense)?

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Example. Let $F : \mathbb{R}^{\ell}_{+} \to \mathbb{R}_{+}$ be production function.

Let $B = \{x \in \mathbb{R}^{\ell}_{+} : F(x) \geqslant q''\}$ and $A = \{x \in \mathbb{R}^{\ell}_{+} : F(x) \geqslant q'\}$, where $q'' > q'$.

Then $\text{argmin}_{x \in B}\, p \cdot x$ and $\text{argmin}_{x \in A}\, p \cdot x$ is the conditional factor demand at output q'' and q' respectively, where $p \in \mathbb{R}^{\ell}_{++}$ are the factor prices.

What restriction on F guarantee that factor demand is normal, i.e., increases with output (in some sense)?

Parallelogram Property

Let the set of parameters be a poset $(T, \geqslant)$.

Definition. The correspondence $\Gamma : T \to \mathbb{R}^\ell$ is increasing in the strong set order if for any $t' \geqslant t$, $x \in \Gamma(t)$, $x' \in \Gamma(t')$, we have $x \vee x' \in \Gamma(t')$ and $x \wedge x' \in \Gamma(t)$.

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Definition. $\Gamma : T \to \mathbb{R}^\ell$ satisfies the parallelogram property if for any $t' \geqslant t$ and $x \in \Gamma(t)$, $x' \in \Gamma(t')$, there is $y \in \Gamma(t)$, $y' \in \Gamma(t')$ such that

$$x' \geqslant y,\ y' \geqslant x \ \text{ and } x + x' = y + y'.$$

Note: If $\Gamma : T \to \mathbb{R}^\ell$ is increasing in the strong set order then Γ has the parallelogram property. Choose $y = x \wedge x'$ and $y' = x \vee x'$.

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Note: If $\Gamma : T \to \mathbb{R}^\ell$ is increasing in the strong set order then Γ has the parallelogram property. Choose $y = x \wedge x'$ and $y' = x \vee x'$.

Definition. The correspondence $\Gamma : T \to \mathbb{R}^\ell$ satisfies increasing property if for any $t' \geqslant t$ and $x \in \Gamma(t)$, $x' \in \Gamma(t')$, there is $y \in \Gamma(t)$, $y' \in \Gamma(t')$ such that $x' \geqslant y$ and $y' \geqslant x$.

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Definition. Let correspondence $\Gamma : T \to \mathbb{R}^\ell$ satisfies parallelogram property if for any $t' \geqslant t$ and $x \in \Gamma(t)$, $x' \in \Gamma(t')$, there is $y \in \Gamma(t)$, $y' \in \Gamma(t')$ such that $x' \geqslant y$, $y' \geqslant x$ and

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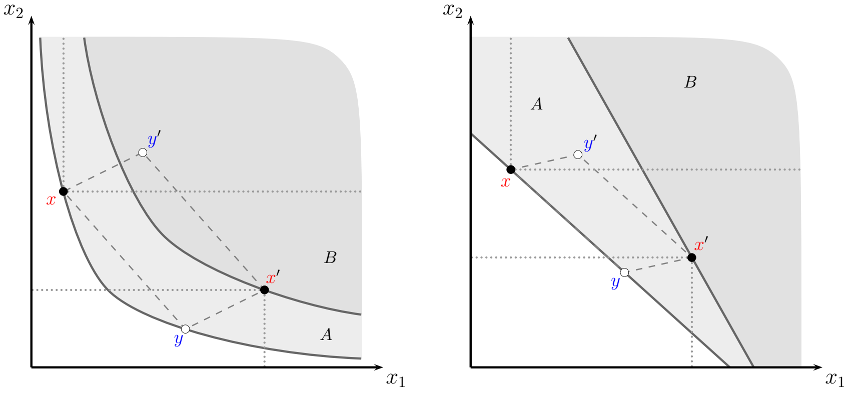

Parallelogram property satisfied on the left but not the right.

Our Basic Result

Main Theorem. Let T be a poset and $\Gamma : T \to \mathbb{R}^\ell$ be a convex-valued correspondence. The following statements are equivalent.

1. The correspondence Γ satisfies parallelogram property.
2. For any $p \in \mathbb{R}^\ell$, the correspondence $\Phi : T \to \mathbb{R}^\ell$, given by

$$\Phi(t) \ := \ \operatorname{argmin}\Big\{p \cdot x : x \in \Gamma(t)\Big\},$$

satisfies the parallelogram property.

3. For any $p \in \mathbb{R}^\ell$, the correspondence Φ satisfies the increasing property.

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Furthermore, suppose Γ is upward comprehensive (so $x \in \Gamma(t)$ implies $x' \in \Gamma(t)$ for any $x' > x$). Then

1 is implied by 3': for any $p \in \mathbb{R}^\ell_{++}$, Φ satisfies the increasing property.

Application 1: Normal factor demand

Let $F : \mathbb{R}^\ell_+ \to \mathbb{R}_+$ be an increasing and quasiconcave production function.

We say that F has the parallelogram property if $\Gamma : \mathbb{R}_+ \to \mathbb{R}^\ell_+$ given by

$$\Gamma(q) = \{x \in \mathbb{R}^\ell_+ : F(x) \geqslant q\}$$

has the parallelogram property.

Immediate application of our Main Theorem:

Γ satisfies parallelogram property if and only if conditional factor demand

$$\Phi(q) \ := \ \operatorname{argmin}\Big\{p \cdot x : x \in \Gamma(q)\Big\},$$

satisfies the parallelogram property at every $p \in \mathbb{R}^\ell_{++}$.

In particular, Φ is increasing with respect to q.

Our Main Result

Proof that if Γ satisfies parallelogram property then so does Φ, where

$$\Phi(t) \ := \ \operatorname{argmin}\Big\{p \cdot x : x \in \Gamma(t)\Big\}.$$

Take any $p \in \mathbb{R}^{\ell}$, $t' \geqslant t$, $x \in \Phi(t)$, and $x' \in \Phi(t')$.

Since $x \in \Gamma(t)$, $x' \in \Gamma(t')$, the parallelogram property on Γ guarantees that there is $y \in \Gamma(t)$ and $y' \in \Gamma(t')$ such that $x + x' = y + y'$ and $x' \geqslant y$, $y' \geqslant x$.

Since $y \in \Gamma(t)$ and $x \in \Phi(t)$, it must be $p \cdot y \geqslant p \cdot x$.

Similarly, $p \cdot y' \geqslant p \cdot x'$. Thus,

$$p \cdot (y + y') \ \geqslant \ p \cdot (x + x') \ = \ p \cdot (y + y'),$$

which holds only if $p \cdot y = p \cdot x$ and $p \cdot y' = p \cdot x'$.

Therefore, $y \in \Phi(t)$ and $y' \in \Phi(t')$. QED

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Our Main Result

The value function $f : \mathbb{R}^\ell \times T \to \mathbb{R}$ is given by

$$f(p,t) := \min\left\{p \cdot y : y \in \Gamma(t)\right\},$$

Definition. The value function f has increasing differences in (p,t) if, for any $t' \geqslant t$,

$$f\left(p,t'\right) - f\left(p,t\right) \text{ is increasing in } p.$$

In the production context, $f(p,q)$ is the cost of producing q.

f has increasing differences means that marginal cost

$$f\left(p,q'\right) - f\left(p,q\right) \text{ increasing with factor prices } p.$$

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satisfies the parallelogram property.
3. For any $p \in \mathbb{R}^\ell$, the correspondence Φ satisfies the increasing property.
4. The value function $f : \mathbb{R}^\ell \times T \to \mathbb{R}$, given by
$$f(p,t) := \min\left\{p \cdot y : y \in \Gamma(t)\right\},$$
has increasing differences in (p, t).

Application 1: Normal Factor Demand and Monotone Marginal Cost

Theorem. Let $F : \mathbb{R}^\ell_+ \to \mathbb{R}_+$ be an increasing and quasiconcave production function. The following statements are equivalent.

1. F satisfies the parallelogram property.

2. For any $p \in \mathbb{R}^\ell_{++}$, the factor demand correspondence $\Phi : \mathbb{R}_+ \to \mathbb{R}^\ell$, given by

$$\Phi(q) \;:=\; \operatorname{argmin}\Big\{p \cdot y : F(y) \geqslant q\Big\}, \tag{1}$$

satisfies the parallelogram property.

3. For any $p \in \mathbb{R}^\ell_{++}$, factor demand Φ satisfies the increasing property.

4. The cost function $f(p,q) := \min\{p \cdot y : F(y) \geqslant q\}$,

has increasing differences.
(In other words, marginal cost increases with p.)

Application 2: Normal Marshallian Demand

Suppose the utility function $u : \mathbb{R}^\ell_+ \to \mathbb{R}$ is increasing and concave.

We say u has the parallelogram property if $\Gamma : \mathbb{R} \to \mathbb{R}^\ell_+$ given by

$\Gamma(t) = \{x \in \mathbb{R}^\ell_+ : u(x) \geqslant t\}$ has the parallelogram property.

Hicksian Demand is $H(t) := \operatorname{argmin}\Big\{p \cdot x : u(x) \geqslant t\Big\}$.

By the previous theorem, we obtain

Theorem. Hicksian Demand satisfies parallelogram property at every $p \in \mathbb{R}^\ell_{++}$ if and only if u satisfies the parallelogram property.

The Marshallian Demand correspondence is

$$D(p, w) = \operatorname{argmax}\{u(x) : x \in B(p, w)\}.$$

Application 2: Normal Marshallian Demand

If utility function is continuous and locally nonsatiated, then

$$D(p, w) = H(p, v(p, w))$$

where $v(p, w) = u(D(p, w))$ is the indirect utility at (p, w).

This identity allows us to translate results from Hicksian Demand to Marshallian Demand.

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Theorem. Let utility $u : \mathbb{R}^\ell_+ \to \mathbb{R}$ be continuous, increasing and quasiconcave. The following statements are equivalent.

1. u satisfies the parallelogram property.

2. For any price $p \in \mathbb{R}^\ell_{++}$, the Marshallian demand correspondence $D(p, \cdot) : \mathbb{R}_+ \to \mathbb{R}^\ell_+$ satisfies the parallelogram property.

3. There is a function $d : \mathbb{R}^\ell_{++} \times \mathbb{R}_+ \to \mathbb{R}$ such that $d(p, w) \in D(p, w)$, for all (p, w), and $d(p, w') \geqslant d_(p, w)$, for all p and $w' \geqslant w$.

Functions satisfying parallelogram property

What functions $F : \mathbb{R}_+^\ell \to \mathbb{R}$ satisfy the parallelogram property?

1. F is homothetic/homogeneous of degree $k > 0$.
2. F is supermodular and concave; for example,
$F(x_1, x_2) = \sqrt{x_1 x_2} + \ln x_2 + x_1$.

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3. Suppose $f_k : \mathbb{R}^{\ell_k} \to \mathbb{R}$ satisfies the parallelogram property for each $k = 1, 2, \ldots, n$ and let G be a supermodular and concave function. Then

$$F(x_1, x_2, \ldots, x_n) = G(f_1(x_1), f_2(x_2), \ldots, f_n(x_n))$$

satisfies the parallelogram property.

4. Let $g : \mathbb{R}_+^2 \to \mathbb{R}$ be concave and supermodular. Then

$$F(x_1, x_2, x_3, x_4) = g(x_4(g(x_3, g(x_2, x_1))))$$

satisfies the parallelogram property. For example,

$$F(x_1, x_2, x_3) = \sqrt{x_1} + \sqrt{\sqrt{x_2} + \sqrt{x_3}}.$$

First order stochastic dominance: the EU case

Let $\mathcal{F}$ be the collection of distributions on $S \subset \mathbb{R}$

Let $(T, \geqslant)$ be set of parameters and let $\lambda : T \to \mathcal{F}$ be a function.

Definition: λ is increasing in first order stochastic dominance if $\lambda(t') \leqslant \lambda(t)$ whenever $t' > t$.

Basic Result 1: λ is FSD-increasing if and only if

$$\int_S g(s)\, d\lambda(s, t') \geqslant \int_S g(s) d\lambda(s, t)$$

for all increasing functions $g : S \to \mathbb{R}$ and $t' > t$.

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Let $\Lambda : T \to \mathcal{F}$ be a convex-valued correspondence. How do we guarantee that

$$\min \left\{ \int_S g(s) d\lambda(s) : \lambda \in \Lambda(t') \right\} \geqslant \min \left\{ \int_S g(s) d\lambda(s) : \lambda \in \Lambda(t) \right\}$$

for all increasing functions $g : S \to \mathbb{R}$ and $t' > t$?

First order stochastic dominance: the MEU case

Definition: Let $\Lambda : T \to \mathcal{F}$ be a correspondence.

Λ is FSD-increasing if, for all $t' > t$, the following holds:

for all $\lambda' \in \Lambda(t')$ there is $\lambda \in \Lambda(t)$ such that $\lambda' \gtrsim_{FSD} \lambda$.

Theorem: The function

$$G(t) = \min \left\{ \int_S g(s) d\lambda(s) : \lambda \in \Lambda(t) \right\}$$

is increasing in t for increasing functions $g : S \to \mathbb{R}$ if and only if $\Lambda : T \to \mathbb{R}$ is FSD-increasing.

Example: $\Lambda(t') \subseteq \Lambda(t)$ whenever $t' > t$.

FSD for comparative statics: the EU case

An agent chooses action $x \in X \subset \mathbb{R}$ under uncertainty to maximize

$$v(x,t) = \int_S u(x,s) d\lambda(s,t)$$

u is supermodular if $u(x'', s) - u(x', s)$ is increasing in s for all $x'' > x'$.

Basic result 2: The function v is supermodular in (x, t) if

(i) $u(x, s)$ is supermodular and

(ii) $\lambda(\cdot, t') \succeq_{FSD} \lambda(\cdot, t)$ if $t' > t$.

Interpretation: the supermodularity of u guarantees that $\arg\max_{x \in X} u(x, s)$ is increasing in s (Milgrom-Shannon Theorem).

If λ is FSD-increasing, then $\arg\max_{x \in X} v(x, t)$ is increasing in t.

Changing stochastic environments

Proof: $\Delta(t) := v(x'', t) - v(x', t) = \int [u(x'', s) - u(x', s)] d\lambda(s, t)$.

If $x'' > x'$, then $\delta(s) = u(x'', s) - u(x', s)$ is increasing in s.

So Δ is increasing in t if λ is FSD-increasing. QED

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Example: An agent lives for two periods.
Income today is w_1 and tomorrow's income s is stochastic.

The expected utility of saving $x \in [0, w_1]$ is

$$v(x, t) = \int_S [u_1(w_1 - x) + \beta u_2(Rx + s)] \, d\lambda(s, t).$$

If u_2 is concave, $(x, s) \to u_1(w_1 - x) + \beta u_2(Rx + s)$ is submodular .

Assuming this, if λ is FSD-increasing, then v is submodular

and hence $\arg\max_{x \in [0, w_1]} v(x, t)$ decreases with t.

FSD for comparative statics: the MEU case

If the agent is ambiguity averse, his objective function is

$$v(x,t) = \min\left\{\int_S u(x,s)d\lambda(s) : \lambda \in \Lambda(t)\right\}.$$

What set-generalization of an FSD shift will guarantee the supermodularity of v?

The property on Λ needed for comparative statics is different from the one needed to compare utilities.

FSD for comparative statics: the MEU case

A possible condition: $\Lambda(t')$ dominates $\Lambda(t)$ if every distribution in $\Lambda(t')$ dominates every distribution $\Lambda(t)$.

FSD for comparative statics: the MEU case

A possible condition: $\Lambda(t')$ dominates $\Lambda(t)$ if every distribution in $\Lambda(t')$ dominates every distribution $\Lambda(t)$.

Choose $x' > x$ and suppose

$v(x,t) = \int u(x,s)d\hat{\lambda}(s)$ for some $\hat{\lambda} \in \Lambda(t)$ and

$v(x',t') = \int u(x',s)d\tilde{\lambda}(s)$ for some $\tilde{\lambda} \in \Lambda(t')$.

Note that $v(x,t') \leqslant \int u(x',s)d\tilde{\lambda}(s)$ and $v(x',t) \leqslant \int u(x,s)d\hat{\lambda}(s)$.

Since $\tilde{\lambda} \gtrsim_{FSD} \hat{\lambda}$ and $x' > x$, we obtain

$$\begin{aligned} v(x',t') - v(x,t') &\geqslant \int \left[u(x',s) - u(x,s)\right] d\tilde{\lambda}(s) \\ &\geqslant \int \left[u(x',s) - u(x,s)\right] d\hat{\lambda}(s) \\ &\geqslant v(x',t) - v(x,t). \end{aligned}$$

Application 3: maxmin expected utility

What assumption on $\Lambda : T \to \mathcal{F}$ will guarantee the supermodularity of

$$v(x,t) = \min \left\{ \int_S u(x,s) d\lambda(s) : \lambda \in \Lambda(t) \right\} ?$$

Definition: $\Lambda : T \to \mathbb{R}$ is strongly FSD-increasing if, for $t' \geqslant t$, $\lambda' \in \Lambda(t')$, and $\lambda \in \Lambda(t)$, there is some $\mu' \in \Lambda(t')$ and $\mu \in \Lambda(t)$ such that

$$\lambda' \succeq_{FSD} \mu, \ \mu' \succeq_{FSD} \lambda, \text{ and } \tfrac{1}{2}\lambda' + \tfrac{1}{2}\lambda = \tfrac{1}{2}\mu' + \tfrac{1}{2}\mu.$$

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Let $S = \{s_i\}_{i=1}^{\ell+1}$ such that $s_1 < \ldots < s_{\ell+1}$.

Then λ can be thought of as the vector $(\lambda(s_1), \lambda(s_2), \ldots, \lambda(s_\ell)) \in \mathbb{R}^\ell$.

Then $\Lambda : T \to \mathbb{R}$ is strongly FSD-increasing if and only if $-\Lambda$ satisfies the parallelogram property.

Application 3: maxmin expected utility

Theorem: Let X and T be subsets of $\mathbb{R}$. The function $v : X \times T \to \mathbb{R}$ given by

$$v(x,t) = \min \left\{ \int_S u(x,s) d\lambda(s) : \lambda \in \Lambda(t) \right\},$$

is supermodular in (x,t) for all functions u which are supermodular in (x,s) if and only if Λ is strongly FSD-increasing.

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Note: If v is supermodular, then $\arg\max_{x\in X} v(x,t)$ is increasing in t.

Proof of sufficiency. For any distribution λ on $S = \{s_1, s_2, \ldots, s_{\ell+1}\}$,

$$\int u(x,s)d\lambda(s) = u(x, s_{\ell+1}) + \sum_{i=1}^{\ell}\left[u(x,s_{i+1}) - u(x,s_i)\right]\left[-\lambda(s_i)\right].$$

Application 3: maxmin expected utility

Therefore, $v(x,t)$ equals

$$u(x,s_{\ell+1}) + \min\left\{\sum_{i=1}^{\ell}\left[u(x,s_{i+1}) - u(x,s_i)\right]\left[-\lambda(s_i)\right] : \lambda \in \Lambda(t)\right\}$$

and v is supermodular iff $f(x,t) = \min\left\{\sum_{i=1}^{\ell} p_i\left[-\lambda(s_i)\right] : \lambda \in \Lambda(t)\right\}$

is supermodular, where $p_i = u(x,s_{i+1}) - u(x,s_i)$.

Application 3: maxmin expected utility

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If $x' > x$, then since u is supermodular,

$p_i' = u(x', s_{i+1}) - u(x', s_i) \geqslant p_i = u(x, s_{i+1}) - u(x, s_i)$ for $i = 1, 2, \ldots, \ell$.

Application 3: maxmin expected utility

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Thus, $f(x',t) - f(x,t) =$
$\min\left\{\sum_{i=1}^{\ell} p_i'\left[-\lambda(s_i)\right] : \lambda \in \Lambda(t)\right\} - \min\left\{\sum_{i=1}^{\ell} p_i\left[-\lambda(s_i)\right] : \lambda \in \Lambda(t)\right\}$

By Main Theorem, $f(x',t) - f(x,t)$ is increasing in t if $-\Lambda$ satisfies the parallelogram property. QED

Application 3: maxmin expected utility

Example (precautionary savings)

A consumer lives for two periods.

Income today is w_1 and tomorrow's income s is stochastic.

The utility of saving $x \in [0, w_1]$ is

$$v(x,t) = \min\left\{\int_S \left[u_1(w_1 - x) + \beta u_2(Rx + s)\right] d\lambda(s) : \lambda \in \Lambda(t)\right\}.$$

If u_2 is concave, then $(x,s) \to u_1(w_1 - x) + \beta u_2(Rx + s)$ is submodular .

Assuming this, if Λ is strongly FSD-increasing in t, then $v(x,t)$ is submodular and hence, $\arg\max_{x \in [0,w_1]} v(x,t)$ is decreasing in t, i.e.,

if high income is more likely tomorrow, the agent saves less today.

Application 3: maxmin expected utility

Examples of strongly FSD-increasing correspondences.

Example 1: Λ is strongly FSD-increasing if it is increasing in the strong set order, i.e.,

for any $\lambda \in \Lambda(t)$ and $\lambda' \in \Lambda(t')$,

$$\lambda \vee \lambda' \in \Lambda(t') \text{ and } \lambda \wedge \lambda' \in \Lambda(t).$$

Specific instance:

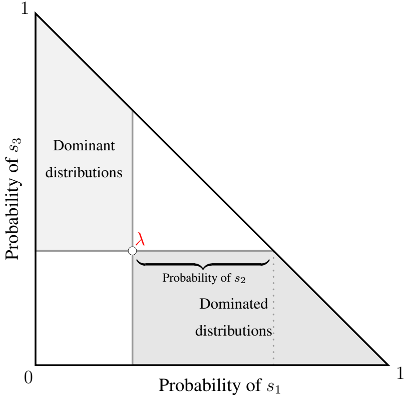
$$\Lambda(t) = [\underline{\theta}(t), \bar{\theta}(t)]$$

where $\bar{\theta}(t) \geq_{FSD} \underline{\theta}(t)$ and both $\bar{\theta}$ and $\underline{\theta}$ are FSD-increasing.

Application 3: maxmin expected utility

Example 2: $\Lambda(t) =$ All distributions on $\mathcal{S}$ with mean t.

Illustration when $s_1 < s_2 < s_3$.



Extension to α-maxmin preferences

In fact, applying the Main Theorem we could also show that

$$w(x,t) = \max\left\{\int_S u(x,s)d\lambda(s) : \lambda \in \Lambda(t)\right\}$$

is a supermodular function when Λ is strongly FSD-increasing,

just as

$$v(x,t) = \min\left\{\int_S u(x,s)d\lambda(s) : \lambda \in \Lambda(t)\right\}$$

is a supermodular function when Λ is strongly FSD-increasing.

Therefore, for any $\alpha \in [0,1]$, the function

$$h(x,t) = \alpha v(x,t) + (1-\alpha)w(x,t)$$

is also supermodular. Hence, $\text{argmax}_{x \in X} h(x,t)$ increases with t.

Application 4: Variational Preferences

The agent's utility from choosing $x \in \mathbb{R}$ is

$$v(x,t) = \min \left\{ \int_S u(x,s) d\lambda(s) + c(\lambda, t) : \lambda \in \triangle_S \right\}$$

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Theorem: $v(x,t)$ is supermodular in (x,t) if $u(x,s)$ is supermodular and c satisfies the following condition $(\star)$:

for any $t' \geqslant t$ and distributions λ' and λ, there is μ' and μ such that

$$\lambda' \succeq_{FSD} \mu, \;\; \mu' \succeq_{FSD} \lambda, \;\; \tfrac{1}{2}\lambda' + \tfrac{1}{2}\lambda = \tfrac{1}{2}\mu' + \tfrac{1}{2}\mu \;\text{ and}$$

$$c(\lambda, t) + c(\lambda', t') \geqslant c(\mu, t) + c(\mu', t').$$

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$$c(\lambda,t) + c(\lambda',t') \geqslant c(\mu,t) + c(\mu',t').$$

It suffices for c to be submodular in λ (for any fixed t):

$$c(\lambda,t) + c(\lambda',t) \leqslant c(\lambda \wedge \lambda',t) + c(\lambda \vee \lambda',t)$$

and have increasing differences:

if $\lambda' \succeq_{FSD} \lambda$, then $c(\lambda',t) - c(\lambda,t)$ increases with t.

Application 5: Multiplier preferences

This is the special case of variational preferences where

$$c(\lambda, t) := \theta R\big(\lambda \| \lambda^*(\cdot, t)\big),$$

for some $\lambda^*(\cdot, t) \in \triangle_S$, where R is the relative entropy, i.e.,

$$R(\lambda \| \lambda^*(\cdot, t)) := \sum_{s \in S} \pi_s \ln\left(\frac{\pi_s}{\pi_s^*(t)}\right)$$

Note: π_s is the probability of state s in the distribution λ.

$\lambda^*(\cdot, t)$ is the reference or benchmark distribution.

$R(\lambda \| \lambda^*(\cdot, t)) = 0$ if $\lambda = \lambda^*$ and is positive otherwise.

[Sargent and Hansen (2001), Strzalecki (2011)]

Application 5: Multiplier preferences

Proposition: For any fixed $\lambda^*(\cdot, t)$, the relative entropy

$$R(\lambda \| \lambda^*(\cdot, t)) := \sum_{s \in S} \pi_s \ln \left(\frac{\pi_s}{\pi_s^*(t)} \right)$$

is a submodular function of $\lambda \in \Delta_S$.

Furthermore, R has increasing differences if $\lambda^*(\cdot, t)$ is increasing in t with respect to the monotone likelihood ratio order, i.e.,

if $t'' > t'$, then the ratio $\pi_s^*(t'')/\pi_s^*(t')$ is increasing with s.

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Recap: $v(x, t) = \min \left\{ \int_S u(x, s) d\lambda(s) + \theta R(\lambda \| \lambda^*(t)) : \lambda \in \triangle_S \right\}$

is supermodular in (x, t) if

(1) u is supermodular in (x, s) and

(2) λ^* is increasing in t with respect to the monotone likelihood ratio.

Application 6: MEU dynamic optimization

The firm's profit in period t is $\pi(x_t, s_t)$, where x_t is the capital stock at the beginning of the period and s_t is the state of the world in period t.

At each period t, a firm decides on the next period's capital stock. The dividend at time t, net of investment is

$$r(x_t, x_{t+1}, s_t) = \pi(x_t, s_t) - c(x_{t+1} - \rho x_t)$$

where c is the cost of investment and ρ is the depreciation rate.

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If we assume that π is supermodular and c is convex, then r is supermodular, over all three arguments.

At the point when x_{t+1} is decided, the firm knows s_t but not s_{t+1}.

$\Lambda(s_t)$ gives the set of distributions on S, conditional on s_t.
Assume that Λ is strongly FSD-increasing.

Application 6: MEU dynamic optimization

With appropriate ancillary assumptions, the firm's decision at time t is governed by the Bellman equation

$$w(x,s) = \max_{y \in \mathbb{R}_+} \left[r(x,y,s) + \min \left\{ \int_S w(y,s') d\lambda(s') : \lambda \in \Lambda(s) \right\} \right],$$

where $w(x,s)$ is the firm's value at (x,s).

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Claim: w is a supermodular function.

Proof: For *any* supermodular function $g(y,s')$, we know from our theorem that

$$\min \left\{ \int_S g(y,s') d\lambda(s') : \lambda \in \Lambda(s) \right\}$$

is a supermodular function in (y,s). Consequently

$$r(x,y,s) + \min \left\{ \int_S g(y,s') d\lambda(s') : \lambda \in \Lambda(s) \right\}$$

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is supermodular in (x, y, s). It follows that

$$(Tg)(x, s) = \max_{y \in \mathbb{R}_+}\left[r(x, y, s) + \min\left\{\int_S g(y, s')d\lambda(s') : \lambda \in \Lambda(s)\right\}\right]$$

is a supermodular function of (x, s). The map T takes one supermodular function to another. T has a fixed point w, where $w(x, s)$ is the firm's value at (x, s).

w is supermodular in (x, s). QED

Application 6: MEU dynamic optimization

The firm's decision at time t is governed by the Bellman equation

$$w(x, s) = \max_{y \in \mathbb{R}_+} \left[r(x, y, s) + \delta \min \left\{ \int_S w(y, s') d\lambda(s') : \lambda \in \Lambda(s) \right\} \right],$$

where $w(x, s)$ is the firm's value at (x, s).

The supermodularity of the objective function implies that the optimal y is increasing in (x, s).

In other words, the firm's choice of capital stock x_{t+1} is increasing in (x_t, s_t).

Conclusion

We develop a basic result on monotone comparative statics

for linear objective functions.

We use it to establish a threefold equivalence:

(1) monotone marginal costs
(2) normal demand
(3) the parallelogram property

We develop a notion of multi-prior first order stochastic dominance

that is necessary and sufficient for monotone comparative statics.

Notes

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