Revealed Price Preference: Theory and Stochastic Testing

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Bounded Rationality in Choice

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Introduction: Revealed preference – the familiar theory

We observe a consumer's demand over some set of L goods.
At observation t, the prevailing prices are

$$p^t = (p_1^t, p_2^t, \ldots, p_L^t) \in \mathbb{R}_{++}^L$$

and the consumer purchases the bundle

$$x^t = (x_1^t, x_2^t, \ldots, x_L^t) \in \mathbb{R}_{+}^L.$$

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Formally, we have access to a data set $\mathcal{D} := \{(p^t, x^t)\}_{t=1}^T$.

We say the consumer *revealed weakly prefers (strictly prefers)* the bundle x^t to the bundle $x^{t'}$, with notation $x^t \succeq_x (\succ_x)\, x^{t'}$, if

$$p^t x^t \geq (>)\, p^t x^{t'}.$$

$\mathcal{D}$ satisfies the generalized axiom of revealed preference (GARP) if $\succeq_x$ has no cycles containing $\succ_x$.

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Definition: A utility function $\tilde{U} : \mathbb{R}^L_+ \to \mathbb{R}$ *rationalizes* $\mathcal{D} = \{(p^t, x^t)\}_{t=1}^T$ if, for all $t = 1, 2, \ldots, T$,

$$x^t \in \operatorname{argmax}\{\tilde{U}(x) : p^t x \leq p^t x^t\}.$$

Afriat's Theorem: Given a data set $\mathcal{D} = \{(p^t, x^t)\}_{t=1}^T$, the following are equivalent:

1. $\mathcal{D}$ can be rationalized by a locally nonsatiated preference
2. $\mathcal{D}$ satisfies GARP.
3. $\mathcal{D}$ can be rationalized by a strictly increasing, continuous, and concave utility function.

Introduction: Revealed preference – the familiar theory

Limitations of this approach:

Typically, we only observe the consumer's demand over a fraction of all goods.

Expenditure on the L observed goods is *endogenous* (contrary to standard textbook presentations of consumer demand).

Utility is defined over *all* goods and not just the L goods observed.

Therefore, empirical work requires a utility function of the form

$$V(U(x), y_1, y_2, y_3, \cdots)$$

where there is a subutility $U(x)$ over L goods, which is 'detachable' (and thus definable) from all other goods.

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Lastly, this model does not tell us whether a subject is better under one set of prices (on the L goods) versus another set of prices.

In fact, scalar multiples of prices are indistinguishable in this model.

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Definition: Given a data set $\mathcal{D} = \{(p^t, x^t)\}_{t=1}^T$, the consumer *revealed weakly prefers (strictly prefers)* p^t to $p^{t'}$ if

$$p^t x^{t'} \leq (<)\, p^{t'} x^{t'}.$$

We use the notation $p^t \succeq_p (\succ_p)\, p^{t'}$.

Motivation:

At price vector p^t, the consumer can buy the bundle bought at observation t' and it will cost him less.

So he must prefer the price p^t to the price $p^{t'}$.

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$\mathcal{D}$ satisfies the generalized axiom of price preference (GAPP) if $\succeq_p$ has no cycles containing $\succ_p$, i.e., if there are observations t_1, $t_2, \ldots, t_n$ such that

$$p^{t_1} \succeq_p p^{t_2},\ p^{t_2} \succeq_p p^{t_3},\ \ldots\ p^{t_{n-1}} \succeq_p p^{t_n}, \text{ and } p^{t_n} \succeq_p p^{t_1}$$

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What optimizing model is observationally equivalent to GAPP?

Expenditure-Augmented Utility

Suppose that a consumer's purchasing behavior over L goods is guided by **(1)** benefit he derives from the L goods and **(2)** disutility of the expenditure incurred from spending money on those goods.

The consumer has an expenditure-augmented utility function (or simply, an augmented utility function) $U : \mathbb{R}^L_+ \times \mathbb{R}_- \to \mathbb{R}$.

If consuming bundle x incurs expenditure e, then the utility is $U(x, -e)$.

In other words, the cost of a bundle is an *attribute* directly affecting its utility.

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Note: This model generalizes the quasilinear utility model where x is chosen to maximize $V(x) - px$.

Rationalization

Definition: A expenditure-augmented utility function

$$U : \mathbb{R}^L_+ \times \mathbb{R}_- \to \mathbb{R}$$

rationalizes $\mathcal{D} = \{(p^t, x^t)\}_{t=1}^T$ if, for all $t = 1, 2, \ldots, T$,

$$x^t \in \operatorname{argmax} \left\{ U(x, -p^t x) : x \in \mathbb{R}^L_+ \right\}.$$

GAPP is also Sufficient for Rationalization

Theorem 1: Given a data set $\mathcal{D} = \{(p^t, x^t)\}_{t=1}^T$, the following are equivalent:

1. $\mathcal{D}$ can be rationalized by an expenditure-augmented preference.
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1. $\mathcal{D}$ can be rationalized by an expenditure-augmented preference.
2. $\mathcal{D}$ satisfies GAPP.
3. $\mathcal{D}$ can be rationalized by an augmented utility function U that is strictly increasing, continuous, and concave.

   Moreover, U is such that $\max_{x \in \mathbb{R}^L_+} U(x, -p \cdot x)$ has a solution for all $p \in \mathbb{R}^L_{++}$.

Necessity of GAPP

The indirect utility at price p (corresponding to U) is

$$V(p) := \max_{x \in \mathbb{R}^L_+} U(x, -px). \tag{1}$$

The consumer *revealed weakly prefers (strictly prefers)* p^t to $p^{t'}$ [notation $p^t \succeq_p (\succ_p)\, p^{t'}$] if $p^t x^{t'} \leq (<)\, p^{t'} x^{t'}$.

If consumer is maximizing $U(x, -px)$, then $p^t \succeq_p (\succ_p)\, p^{t'} \implies$

$$V(p^t) \geq U(x^{t'}, -p^t x^{t'}) \geq (>)\, U(x^{t'}, -p^{t'} x^{t'}) = V(p^{t'}).$$

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Suppose there are observations $t_1,\ t_2, \ldots, t_n$ such that

$$p^{t_1} \succeq_p p^{t_2},\ p^{t_1} \succeq_p p^{t_2},\ .\ .\ .\ p^{t_{n-1}} \succeq_p p^{t_n}, \text{ and } p^{t_n} \succeq_p p^{t_1}$$

Then we obtain $V(p^{t_1}) \geq V(p^{t_2}) \geq \ldots V(p^{t_n}) \geq V(p^{t_1})$, which means we cannot replace $\succeq_p$ with $\succ_p$ anywhere in the cycle.

Features of the revealed price preference model

It does not require the assumption of separability of observed goods, i.e., does not require $V(U(x), y_1, y_2, ...)$.

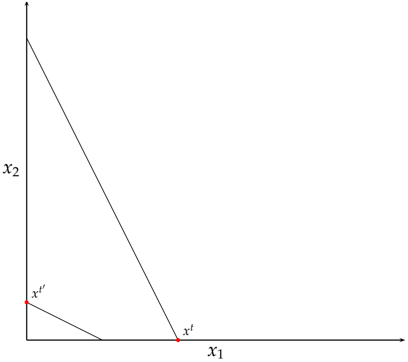
But it does require that outside prices are constant or at least that their changes can be tracked by a composite price index of non-observed goods.

It allows for consistent welfare comparisons between price vectors. (Indeed, the whole theory is built on this consistency.)

It is readily generalized to a *realistic* random utility framework (unlike the standard model).

Data Satisfies GARP but not GAPP

The data set with $p^t = (2, 1)$, $x^t = (4, 0)$ and $p^{t'} = (1, 2)$, $x^{t'} = (0, 1)$ satisfies GARP.



But notice that $p^t x^t = 8 > p^{t'} x^t = 4$, so $p^{t'} \succ_p p^t$, and

$p^{t'} x^{t'} = 2 > p^t x^{t'} = 1$, so $p^t \succ_p p^{t'}$. Data fails GAPP.

Testing GAPP vs. GARP

Scaling Proposition: Let $\mathcal{D} = \{(p^t, x^t)\}_{t=1}^T$ be a data set and let $\widetilde{\mathcal{D}} = \{(p^t, \tilde{x}^t)\}_{t \in T}$, where $\tilde{x}^t = x^t/(p^t x^t)$, be its expenditure-normalized version.

Then the revealed preference relations are related as follows:

1. $p^t \succeq_p p^{t'}$ if and only if $\tilde{x}^t \succeq_x \tilde{x}^{t'}$.
2. $p^t \succ_p p^{t'}$ if and only if $\tilde{x}^t \succ_x \tilde{x}^{t'}$.

Proof: $p^t \frac{x^t}{p^t x^t} \geq p^t \frac{x^{t'}}{p^{t'} x^{t'}}$ iff $p^{t'} x^{t'} \geq p^t x^{t'}$. □

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This implies that testing GAPP on $\mathcal{D}$ is equivalent to scaling the consumption bundles and testing GARP on $\widetilde{\mathcal{D}}$.

Data Satisfies GARP but not GAPP

The data set:

$$p^t = (2, 1)\,,\; x^t = (4, 0) \;\text{ and }\; p^{t'} = (1, 2)\,,\; x^{t'} = (0, 1)$$

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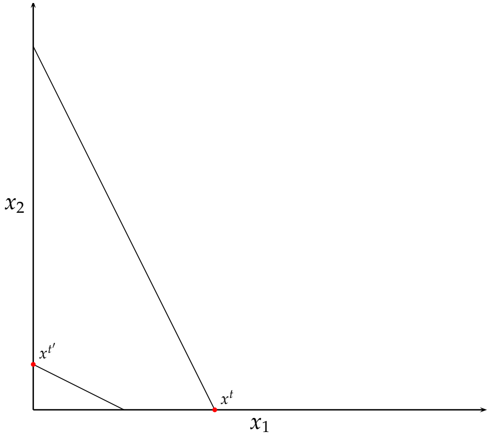

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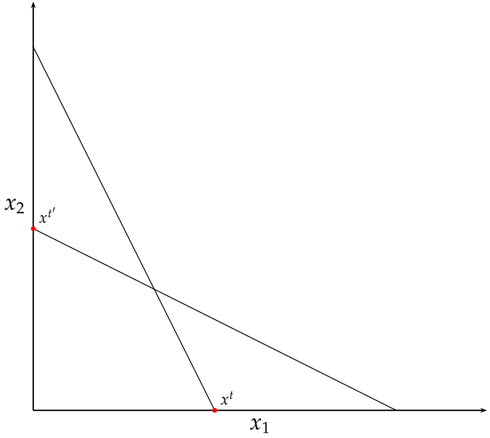
Scaling bundles, it is clear that

$$p^t = (2,1)\,,\; x^t = (4,0) \;\text{ and }\; p^{t'} = (1,2)\,,\; x^{t'} = (0,1)$$

violates GAPP. Scaled bundles: $\tilde{x}^t = (4,0)$. $\tilde{x}^{t'} = (0,4)$.



Welfare Analysis

Rationalizable data $\mathcal{D}$ could potentially be rationalized by many U's leading to different indirect utilities V's.

Notation:

- $\mathbf{V}(\mathcal{D})$: set of rationalizing indirect utility functions.
- $\succeq_p^*$ ($\succ_p^*$): transitive closure of $\succeq_p$ (with one relation strict).

Welfare Proposition: Suppose $\mathcal{D} = \{(p^t, x^t)\}_{t=1}^T$ is rationalizable by an augmented utility function. Then for any p^t, $p^{t'}$:

1. $p^t \succeq_p^* p^{t'}$ if and only if $V(p^t) \geq V(p^{t'})$ for all $V \in \mathbf{V}(\mathcal{D})$.
2. $p^t \succ_p^* p^{t'}$ if and only if $V(p^t) > V(p^{t'})$ for all $V \in \mathbf{V}(\mathcal{D})$.

Thus, $\succ_p$ allows for welfare comparisons under different prices.

Random utility in the standard model (McFadden-Richter)

In this case, an observation t consists of a distribution of demand bundles at price p^t and income w^t.

With a finite set of such distributions, how do we check whether the observed distributions can be generated by a population of utility-maximizing consumers, with this requirement:

the distribution of utility functions is invariant across observations.

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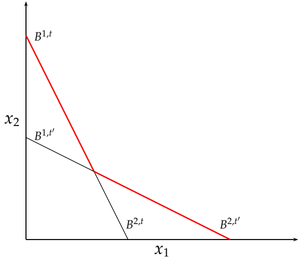
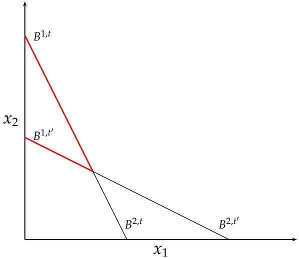
McFadden and Richter's crucial observation:

the number of observationally distinguishable utility-types is *finite*, so this problem is equivalent to solving an appropriate linear program.

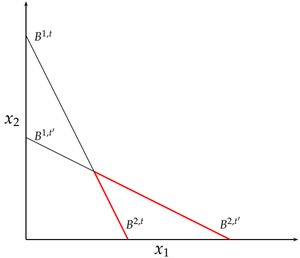
Stochastic GARP



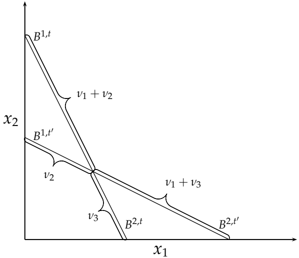
(a) Prop. of this Preference= ν_1



(b) Prop. of this Preference= ν_2

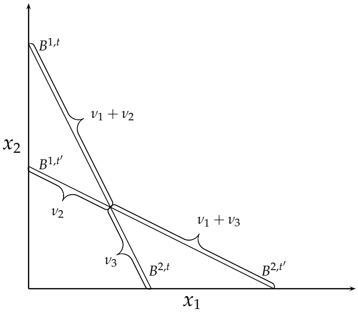


(c) Prop. of this Preference= ν_3



(d) Choice Distribution

Stochastic GARP



(a) Rationalizable Shares (b) Observed Distribution

The observed distribution of choices can be stochastically rationalized if there exist $\nu_1, \nu_2, \nu_3 \geq 0$ such that

$$\nu_1 + \nu_2 = \pi^{1,t},\ \nu_2 = \pi^{1,t'},\ \nu_3 = \pi^{2,t},\ \nu_1 + \nu_3 = \pi^{2,t'}$$

Limitations of the McFadden-Richter model

The problem with the McFadden-Richter model is that data of the type imagined is not actually observed.

Expenditure levels are not exogenously prescribed but *endogenously determined* by the population of consumers.

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Expenditure levels are not exogenously prescribed but *endogenously determined* by the population of consumers.

The data set is $\mathscr{D} := \{(p^t, \tilde{\pi}^t)\}_{t=1}^{T}$. At observation t,

$\tilde{\pi}^t$ is the distribution of demand bundles on $\mathbb{R}_+^L$ —

these bundles *need not* generate the same expenditure at p^t.

A model of choice that rationalizes $\mathscr{D}$ must also explain the observed distribution of expenditure levels.

Rationalization by Random Augmented Utility

Let $\mathscr{D} := \{(p^t, \tilde{\pi}^t)\}_{t=1}^T$.

$\mathscr{D}$ is said to be rationalized by the random augmented utility model if there exists a distribution μ over the set $\mathscr{U}$ of augmented utility functions such that

$$\tilde{\pi}^t(X^t) = \mu(\mathscr{U}(X^t)) \text{ for all } t \in T \text{ and } X^t \subset \mathbb{R}^L_+,$$

where

$$\mathscr{U}(X^t) := \left\{ U \in \mathscr{U} \;:\; \underset{x \in \mathbb{R}^L_+}{\operatorname{argmax}}\, U(x, -p^t x) \in X^t \right\}.$$

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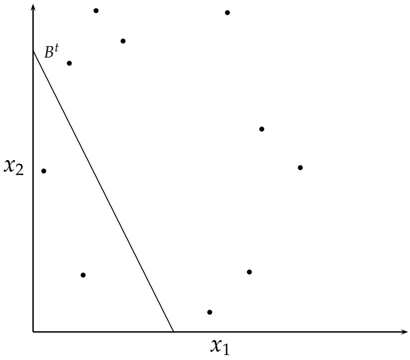
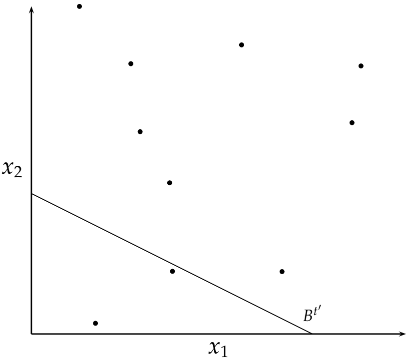
Crucial observation:

By the Scaling Proposition, testing this model is the *same* as testing the McFadden-Richter model after suitable scaling of demand bundles.

Observed Choices

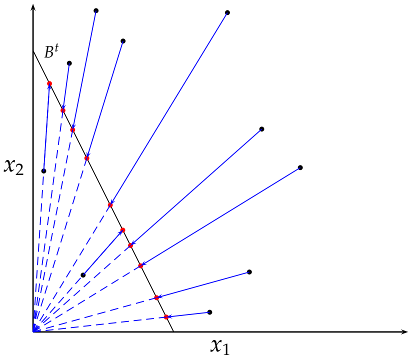


(a) $p^t = (1, 2)$



(b) $p^t = (2, 1)$

Scaled Choices



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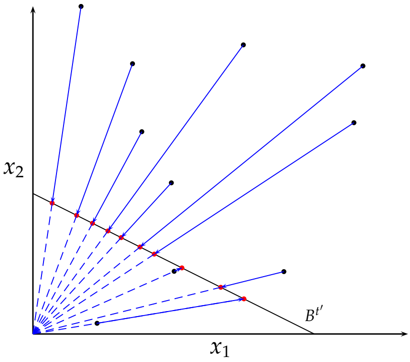

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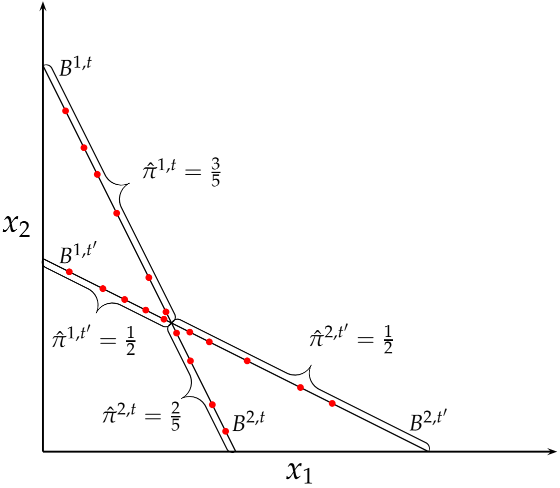
Scaled Empirical Choice Probabilities



Procedure for testing the random augmented utility model

1. Fix an arbitrary expenditure, say 1.
2. At each t, scale each observed choice x^t to $\frac{x^t}{p^t x^t}$.
3. This yields a distribution of choices on T budget planes.
4. Test stochastic GARP on these scaled empirical choice probabilities.

Theorem 2: $\mathscr{D}$ can be rationalized by the random augmented utility model if and only if stochastic GARP holds on the scaled data set.

Concluding Remarks

We develop a simple and intuitive model that allows for consistent welfare comparisons between prices.

This model is easily extendable into a random utility model

We implement the test of the random augmented utility model on UK household spending data:

- The model is not rejected.
- We estimate the proportion of consumers who are better/worse off between one observation and another.

  These estimates are reasonably tight.