# Revealed Price Preference: Theory and Empirical Analysis

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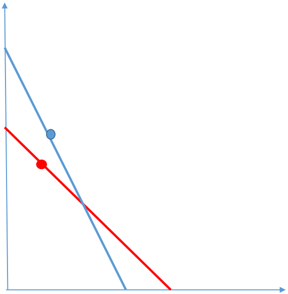
# Example

We observe a consumer's purchases of groceries at the supermarket.

$p^B = (2, 1)$, $x^B = (5, 12)$

$p^R = (1, 1)$, $x^R = (4, 10)$



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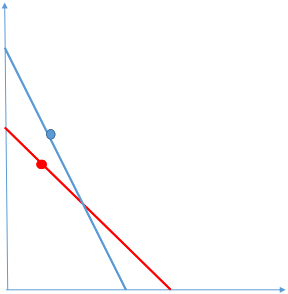

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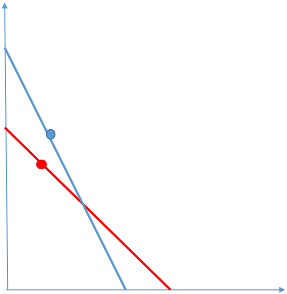

Is the consumer better off at B observation or R observation?

Answer depends on your interpretation of my question.

# Introduction: Revealed preference – the familiar model

We observe a consumer's demand over some set of $L$ goods.
At observation $t$, the prevailing prices are

$$p^t = (p_1^t, p_2^t, \ldots, p_L^t) \in \mathbb{R}^L_{++}$$

and the consumer purchases the bundle

$$x^t = (x_1^t, x_2^t, \ldots, x_L^t) \in \mathbb{R}^L_{+}.$$

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We say the consumer revealed weakly prefers the bundle $x^t$ to the bundle $x^{t'}$, with notation $x^t \succeq_x x^{t'}$, if $p^t x^t \geq p^t x^{t'}$.

The consumer revealed strictly prefers the bundle $x^t$ to the bundle $x^{t'}$, with notation $x^t \succ_x x^{t'}$, if $p^t x^t > p^t x^{t'}$.

$\mathcal{D} = \{(p^t, x^t)\}_{t=1}^{T}$ obeys the generalized axiom of revealed preference (GARP) if $\succeq_x$ has no cycles containing $\succ_x$.

In other words, there are no observations $t_1,\ t_2, \ldots, t_n \in T$ such that

$$x^{t_1} \succeq_x x^{t_2},\ x^{t_2} \succeq_x x^{t_3},\ .\ .\ .\ x^{t_{n-1}} \succeq_x x^{t_n}, \text{ and } x^{t_n} \succeq_x x^{t_1}$$

where one of the relations can be replaced by $\succ_x$.

# Introduction: Revealed preference – the familiar model

Definition: A utility function $\tilde{U} : \mathbb{R}^L_+ \to \mathbb{R}$ *rationalizes* $\mathcal{D} = \{(p^t, x^t)\}_{t=1}^T$ if, for all $t = 1, 2, \ldots, T$,

$$x^t \in \operatorname{argmax}\{\tilde{U}(x) : p^t x \leq p^t x^t\}.$$

Afriat's Theorem: Given a data set $\mathcal{D} = \{(p^t, x^t)\}_{t=1}^T$, the following are equivalent:

1. $\mathcal{D}$ can be rationalized by a locally nonsatiated preference
2. $\mathcal{D}$ satisfies GARP.
3. $\mathcal{D}$ can be rationalized by a strictly increasing, continuous, and concave utility function.

# Introduction: Features of the familiar model

Typically, we only observe the consumer's demand over a fraction of all goods. Expenditure on the $L$ observed goods is *endogenous*.

Utility is defined over *all* goods and not just the $L$ goods observed.

Therefore, empirical work requires a utility function of the form

$$V(\tilde{U}(x), y_1, y_2, y_3, \cdots)$$

where there is a subutility $\tilde{U}(x)$ over $L$ goods, which is 'detachable' (and thus definable) from all other goods.

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where there is a subutility $\tilde{U}(x)$ over $L$ goods, which is 'detachable' (and thus definable) from all other goods.

No 'ceteris paribus' assumption is needed:

the world outside the observed goods do not matter.

Scalar multiples of the $L$ prices are indistinguishable in this model.

But this means the model can say *nothing* about the consumer's preference over prices of the observed goods.

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We use the notation $p^t \succeq_p (\succ_p) \, p^{t'}$ and $p^t \succ_p \ p^{t'}$.

Motivation:

At price vector $p^t$, the consumer can buy the bundle bought at observation $t'$ and it will cost him less.

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So he must prefer the price $p^t$ to the price $p^{t'}$.

Notice that this motivation requires the existence of outside opportunities.

# Inconsistent price preferences

Suppose $p^t = (2,1)$, $x^t = (4,0)$, and $p^{t'} = (1,2)$, $x^{t'} = (0,1)$.

Since $p^t x^t = 8 > p^{t'} x^t = 4$, we have $p^{t'} \succ_p p^t$.

Since $p^{t'} x^{t'} = 2 > p^t x^{t'} = 1$, so $p^t \succ_p p^{t'}$.

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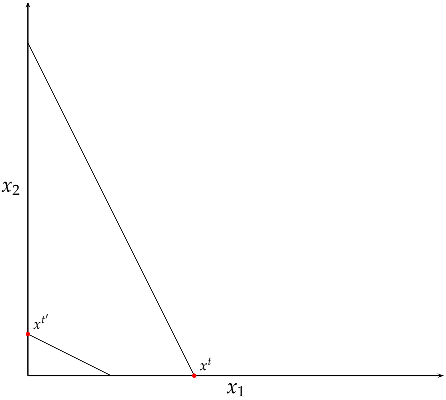
And yet there is no violation of GARP



# Revealed price preference

We need to impose a condition on $\mathcal{D} = \{(p^t, x^t)\}_{t=1}^{T}$ that excludes inconsistent price comparisons.

$\mathcal{D}$ satisfies the generalized axiom of price preference (GAPP) if $\succeq_p$ has no cycles containing $\succ_p$, i.e.,

there are no observations $t_1,\ t_2, \ldots, t_n$ in $T$ such that

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What optimizing model is observationally equivalent to GAPP?

# Expenditure-Augmented Utility

Suppose a consumer's purchasing behavior is guided by

**(1)** benefit he derives from the $L$ goods and

**(2)** disutility of the expenditure incurred from spending money on those goods.

The consumer has an expenditure-augmented utility function

$$U : X \times \mathbb{R}_- \to \mathbb{R},$$

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So the cost of a bundle is an *attribute* directly affecting its utility.

We assume $U(x, -e)$ is *strictly* decreasing in expenditure $e$.

At a price $p$, the consumer chooses $x \in X$ to maximize $U(x, -px)$.

# Rationalization

Definition: A expenditure-augmented utility function

$$U : X \times \mathbb{R}_- \to \mathbb{R}$$

*rationalizes* $\mathcal{D} = \{(p^t, x^t)\}_{t=1}^T$ if, for all $t = 1, 2, \ldots, T$,

$$x^t \in \text{argmax} \left\{ U(x, -p^t x) : x \in X \right\}.$$

# GAPP and rationalization

Theorem 1: Given a data set $\mathcal{D} = \{(p^t, x^t)\}_{t=1}^{T}$, the following are equivalent:

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   Moreover, $U$ is such that $\max_{x \in \mathbb{R}^L_+} U(x, -p \cdot x)$ has a solution for all $p \in \mathbb{R}^L_{++}$.

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**Note:**

$$V(p) = \max_{x \in \mathbb{R}^L_+} U(x, -p \cdot x)$$

gives the agent's preference over prices.

# Necessity of GAPP

The indirect utility at price $p$ (corresponding to $U$) is

$$V(p) := \max_{x \in \mathbb{R}^L_+} U(x, -px). \qquad (1)$$

The consumer *revealed weakly prefers (strictly prefers)* $p^t$ to $p^{t'}$ [notation $p^t \succeq_p (\succ_p)\, p^{t'}$] if $p^t x^{t'} \leq (<)\, p^{t'} x^{t'}$.

If consumer is maximizing $U(x, -px)$, then $p^t \succeq_p (\succ_p)\, p^{t'} \implies$

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Suppose there are observations $t_1, \; t_2, \ldots, t_n$ such that

$$p^{t_1} \succeq_p p^{t_2}, \; p^{t_1} \succeq_p p^{t_2}, \; . \;\; . \;\; . \;\; p^{t_{n-1}} \succeq_p p^{t_n}, \text{ and } p^{t_n} \succeq_p p^{t_1}$$

Then we obtain $V(p^{t_1}) \geq V(p^{t_2}) \geq \ldots V(p^{t_n}) \geq V(p^{t_1})$, which means we cannot replace $\succeq_p$ with $\succ_p$ anywhere in the cycle.

# GAPP: Proof of Sufficiency

Let $M > \max_{t,t'} p^t x^{t'}$. Define the augmented data set

$$\tilde{\mathcal{D}} = \{(p^t, 1), (x^t, M - p^t x^t)\}_{t=1}^{T}$$

where we consider $M$ to be the wealth and the price of the numeraire to be 1.

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$$(p^t, 1)(x^t, M - p^t x^t) \geq (p^t, 1)(x^{t'}, M - p^{t'} x^{t'}) \text{ iff } p^{t'} x^{t'} \geq p^t x^{t'}.$$

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Consequently, $\mathcal{D}$ obeys GAPP if and only if $\tilde{\mathcal{D}}$ obeys GARP.

By Afriat's Theorem, there is a strictly monotone and concave function $\tilde{U}$ such that

$$(x^t, M - p^t x^t) \in \underset{p^t x + m \leq M}{\operatorname{argmax}} \tilde{U}(x, m).$$

Defining $U(x, -e) = \tilde{U}(x, M - e)$ implies that $x^t$ solves $\max_x U(x, -p^t \cdot x)$.

Finally, $U$ can be altered to ensure its maximization always has a solution.

# Welfare Analysis

$\mathcal{D}$ could potentially be rationalized by many $U$'s leading to different indirect utilities $V$'s.

Notation:

- $\mathbf{V}(\mathcal{D})$: set of rationalizing indirect utility functions.
- $\succeq_p^*$: transitive closure of $\succeq_p$.
- $\succ_p^*$: transitive closure of $\succeq_p$, with one relation strict.

Welfare Proposition: Suppose $\mathcal{D} = \{(p^t, x^t)\}_{t=1}^T$ is rationalizable by an augmented utility function. Then for any $p^t$, $p^{t'}$:

1. $p^t \succeq_p^* p^{t'}$ if and only if $V(p^t) \geq V(p^{t'})$ for all $V \in \mathbf{V}(\mathcal{D})$.
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Thus, $\succ_p^*$ and $\succeq_p^*$ are the only *robust* welfare comparisons across the price vectors in $\{p^t\}_{t=1}^T$.

# Features of the GAPP model

This model does not require the existence of a subutility $\tilde{U}$ over observed goods, i.e., does not require $V(\tilde{U}(x), y_1, y_2, ...)$.

A manifestation of this is that a data set can obey GAPP but violate GARP.

Sometimes this could be a useful feature:

the MRS between two goods at a given bundle $x$ can depend on outside consumption.

However, this is *not* the distinctive contribution of the new model.

# Features of the GAPP model

The distinctive contribution is that it allows us to compare the price vectors of observed goods.

To do this, the model *does require* a ceteris paribus assumption.

Or, at least, that we can track external changes in a simple way.

(Otherwise, it is not reasonable to assume that the augmented utility function will remain the same across observations.)

This requirement may not always hold, but when it does, it allows us to answer questions a GARP-based model cannot address.

# Expenditure-Augmented Utility: interpretations

We assume the consumption space $X$ is a closed subset of $\mathbb{R}^L_+$.

Examples: $X = \mathbb{N}^L_+$, $X = \{0, 1\}^L$, or $X = \{e_i\}_{i=1}^L$ (canonical basis).

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Case 1. It generalizes the quasilinear utility model where $x$ is chosen to maximize

$$U(x, -px) = V(x) - px.$$

Versions of this model with $X = \{0, 1\}^L$, or $X = \{e_i\}_{i=1}^L$ are widely used in empirical IO.

A data set $\mathcal{D} = \{(p^t, x^t)\}_{t=1}^T$ generated by such a preference will obey both GARP and GAPP.

The characterization of quasilinearity on $\mathcal{D}$ is known.

(It is strictly stronger than GARP + GAPP.)

# Expenditure-Augmented Utility: interpretations

Case 2. ('Standard' interpretation) Suppose the consumer has total wealth of $M$ which is spent on the $L$ observed goods plus outside goods $y$ at prices $q$. Assume that he maximizes an overall utility function $\tilde{U}$ defined on all goods (observed and unobserved).

Then

$$U(x, -px) := \max_{y \in \mathbb{R}^L_+} \{\tilde{U}(x, y) : px + qy = M\}$$

is an expenditure-augmented utility function.

In other words, keeping unobserved prices $q$ *fixed* and varying $p$, the consumer who maximizes $\tilde{U}$ behaves as though he is maximizing an expenditure-augmented utility function.

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**Note:** The agent's demand for unobserved goods can vary with $p$.

**Note:** A data set $\mathcal{D} = \{(p^t, x^t)\}_{t=1}^T$ collected from such an agent will obey GAPP, but typically not GARP.

# Expenditure-Augmented Utility: interpretations

Case 3. ('Standard' interpretation with changing outside prices)

Assume that the consumer's overall utility function has the form

$$\tilde{U}(x, y) = G(x, h(y))$$

where $h$ is homogeneous of degree one.

The indirect utility of spending $w$ on the outside goods, is $v(q, w) = a(q)w$, where function $a$ is homogeneous of degree -1.

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Agent's budget constraint is $p_1 x_1 + p_2 x_2 + \ldots p_L x_L + w = M/a(q)$, which gives

$$v(q, w) := a(q)w = M - a(q) \left( \sum p_i x_i \right).$$

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$$v(q, w) := a(q)w = M - a(q)\left(\sum p_i x_i\right).$$

Thus the consumes behaves as though he is maximizing

$$U(x, -px, a(q)) = G(x, v(q, w)) = G(x, M - a(q)px).$$

In other words, prices of $L$ goods are deflated by a price index $a(q)$ on the outside goods.

# Expenditure-Augmented Utility: interpretations

Case 4. Why have an 'interpretation'?

It is useful to connect the new concept with something more familiar.

But the budget $M$ is not something directly observable. It may not coincide with anything that is conventionally used (total annual expenditure or annual income).
There could be 'mental budgeting.' (Thaler, 1999)

The new concept is reasonable as it is.

It is not prima facie implausible to *directly* hypothesize that a consumer's behavior over some observed set of goods $L$ is determined by

$$U(x, -px)$$

where utility is derived from the good and disutility from expenditure.

# Expenditure-Augmented Utility: interpretations

Case 4. Why have an 'interpretation'?

It is useful to connect the new concept with something more familiar.

But the budget $M$ is not something directly observable. It may not coincide with anything that is conventionally used (total annual expenditure or annual income).
There could be 'mental budgeting.' (Thaler, 1999)

The new concept is reasonable as it is.

It is not prima facie implausible to *directly* hypothesize that a consumer's behavior over some observed set of goods $L$ is determined by

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where utility is derived from the good and disutility from expenditure.

# Horses for courses

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If we are prepared to make a GAPP hypothesis, then Theorem 1 gives us a test that could either confirm or reject the hypothesis.

And if it is not rejected, we can ask and answer questions a GARP-based model cannot handle.

We apply GAPP in a model of random expenditure augmented utility.

# Testing GAPP by testing GARP

Scaling Proposition: Let $\mathcal{D} = \{(p^t, x^t)\}_{t=1}^T$ be a data set and let $\widetilde{\mathcal{D}} = \{(p^t, \tilde{x}^t)\}_{t \in T}$, where $\tilde{x}^t = x^t/(p^t x^t)$, be its expenditure-normalized version.

Then the revealed preference relations are related as follows:

1. $p^t \succeq_p p^{t'}$ if and only if $\tilde{x}^t \succeq_x \tilde{x}^{t'}$.
2. $p^t \succ_p p^{t'}$ if and only if $\tilde{x}^t \succ_x \tilde{x}^{t'}$.

Proof: $p^t \frac{x^t}{p^t x^t} \geq p^t \frac{x^{t'}}{p^{t'} x^{t'}}$ iff $p^{t'} x^{t'} \geq p^t x^{t'}$. □

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This implies the following:

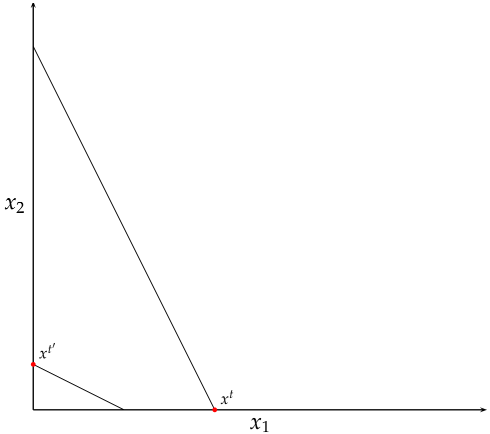
> testing GAPP on $\mathcal{D}$ is equivalent to scaling the consumption bundles and testing GARP on $\widetilde{\mathcal{D}}$.

# Data Satisfies GARP but not GAPP

The data set:

$$p^t = (2,1)\,,\ x^t = (4,0) \ \text{ and } \ p^{t'} = (1,2)\,,\ x^{t'} = (0,1)$$

satisfies GARP.

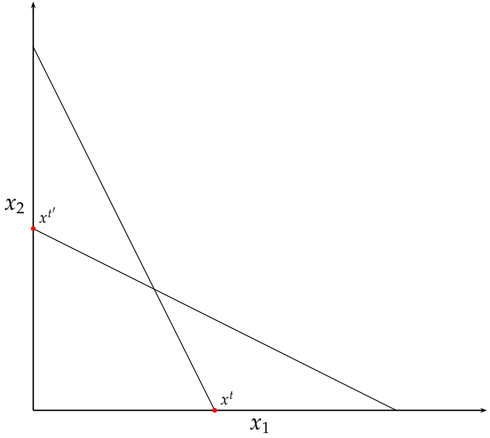


# Data Satisfies GARP but not GAPP

Scaling bundles, it is clear that

$$p^t = (2,1)\,,\; x^t = (4,0) \;\text{ and }\; p^{t'} = (1,2)\,,\; x^{t'} = (0,1)$$

violates GAPP. Scaled bundles: $\tilde{x}^t = (4,0)$. $\tilde{x}^{t'} = (0,4)$.



# Rationalization by Random Augmented Utility

Consider a data set

$$\mathscr{D} := \{(p^t, \tilde{\pi}^t)\}_{t=1}^{T}$$

of the type typically collected in household expenditure surveys.

At the price $p^t$, $\tilde{\pi}^t$ is the distribution of demand bundles on $\mathbb{R}_+^L$ generated by a population of consumers.

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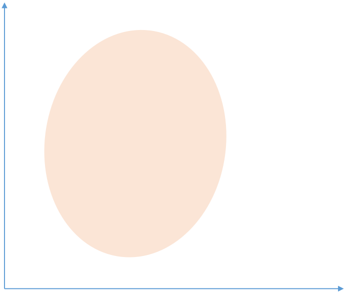

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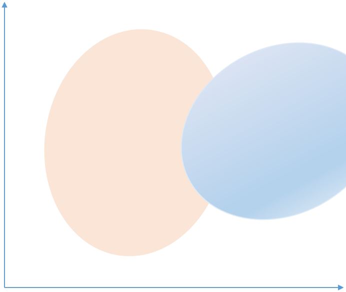

# Rationalization by Random Augmented Utility

Different consumers decide to allocate different expenditures to the $L$ observed goods. So for consumers $i$ and $j$, typically,

$$p^t x_i^t \neq p^t x_j^t$$

(expenditures vary across the population) and typically,

$$p^{t'} x_i^{t'} \neq p^t x_i^t$$

(the same consumer has different total expenditures at different times).

# Rationalization by Random Augmented Utility

Question 1

Can we find a necessary and sufficient condition to guarantee the rationalizability of the data set with a population of consumers maximizing expenditure augmented utility functions?

Question 2

Can we estimate the proportion of the population who are better off at $p^t$ versus $p^{t'}$?

# The Random Augmented Utility model

$\mathscr{D} := \{(p^t, \tilde{\pi}^t)\}_{t=1}^{T}$ is said to be rationalized by the random augmented utility model if there exists a distribution $\mu$ over the set $\mathscr{U}$ of augmented utility functions such that, for any measurable set $\mathcal{S} \subset X \subset \mathbb{R}_+^L$,

$$\tilde{\pi}^t(\mathcal{S}) = \mu(\mathscr{U}^t(\mathcal{S})) \text{ for all } t \in T\text{, where,}$$

$$\mathscr{U}^t(\mathcal{S}) := \left\{ U \in \mathscr{U} \;:\; \operatorname*{argmax}_{x \in \mathbb{R}_+^L} U(x, -p^t x) \in \mathcal{S} \right\}.$$

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If $\mu$ is the true distribution rationalizing data, the proportion of agents who are better off at $p^t$ compared to $p^{t'}$ is given by

$$\mu \left\{ U \in \mathscr{U} \ : \ \operatorname*{argmax}_{x \in \mathbb{R}^L_+} U(x, -p^t x) \geq \operatorname*{argmax}_{x \in \mathbb{R}^L_+} U(x, -p^{t'} x) \right\}.$$

# Rationalizing $\mathscr{D}$ in the GARP-model

The data set is $\mathscr{D} := \{(p^t, \tilde{\pi}^t)\}_{t=1}^T$.

Let $\mathscr{F}$ be the family of functions mapping $\mathbb{R}^L_{++}$ to $R_+$.

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Let $\mathscr{F}$ be the family of functions mapping $\mathbb{R}_{++}^L$ to $R_+$.

Let $\mathscr{V}$ be the family of utility functions mapping $R_+^L$ to $R$.

$\mathscr{D}$ is rationalized if there is a distribution $\nu$ on $\mathscr{V} \times \mathscr{F}$ such that for any measurable set $\mathcal{S} \subset \mathbb{R}_+^L$,

$$\tilde{\pi}^t(\mathcal{S}) = \nu(\mathscr{G}^t(\mathcal{S})) \text{ for all } t \in T, \text{ where}$$

$$\mathscr{G}^t(\mathcal{S}) := \left\{ (V, f) \in \mathscr{V} \times \mathscr{F} \; : \; \underset{x \in B(p^t, f(p^t))}{\operatorname{argmax}} V(x) \in \mathcal{S} \right\}.$$

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McFadden and Richter outlines a way of solving this problem in the special case where expenditure does not vary with price, i.e., where $\mathscr{F}$ is restricted to the function $f(p) = 1$ for all $p \in R_{++}^L$.

Not clear how to deal with the general case.

# Rationalization by Random Augmented Utility

Making use of the McFadden-Richter result (in an unobvious way) we develop a test that answers our two questions.

A1 We can find a necessary and sufficient condition to guarantee the rationalizability of the data set with a population of consumers maximizing expenditure augmented utility functions.

A2 We can estimate what proportion of the population are better off at $p^t$ versus $p^{t'}$?

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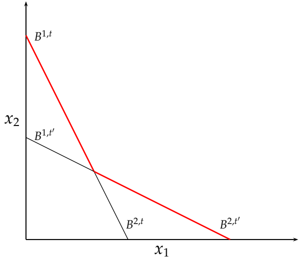
A2 We can estimate what proportion of the population are better off at $p^t$ versus $p^{t'}$?

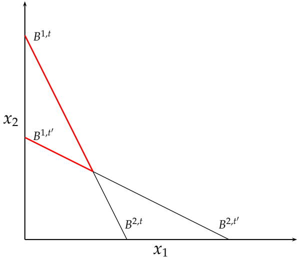
If the data set is ideal, in the sense that we observe true population distributions, then the tests are linear programming problems.

For a real data set, there are econometric techniques that would allow us to deal with sampling issues.

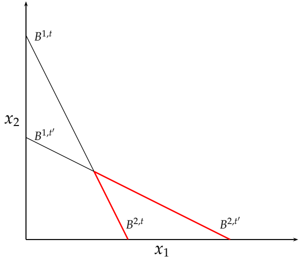
# Stochastic GARP



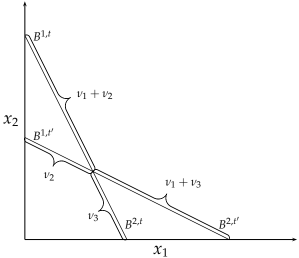

(a) Prop. of this Preference= $\nu_1$




(b) Prop. of this Preference= $\nu_2$

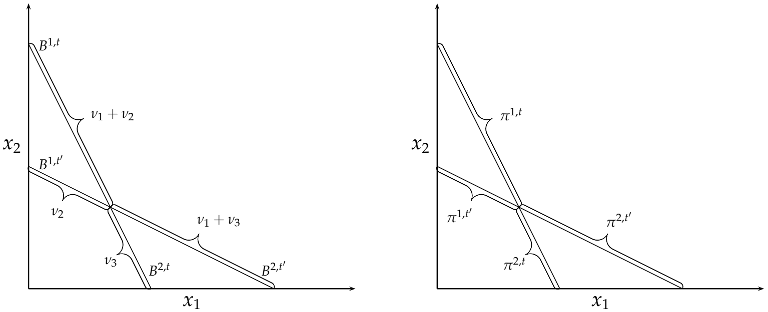



(c) Prop. of this Preference= $\nu_3$




(d) Choice Distribution

# Stochastic GARP




(a) Rationalizable Shares (b) Observed Distribution

The observed distribution of choices can be stochastically rationalized if there exist $\nu_1, \nu_2, \nu_3 \geq 0$ such that

$$\nu_1 + \nu_2 = \pi^{1,t},\ \nu_2 = \pi^{1,t'},\ \nu_3 = \pi^{2,t},\ \nu_1 + \nu_3 = \pi^{2,t'}$$

# Rationalization by Random Augmented Utility

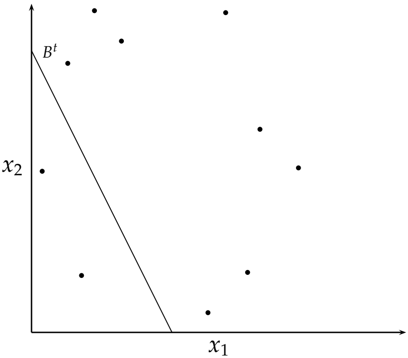
Scaling Proposition: Let $\mathcal{D} = \{(p^t, x^t)\}_{t=1}^{T}$ be a data set and let $\widetilde{\mathcal{D}} = \{(p^t, \tilde{x}^t)\}_{t \in T}$, where $\tilde{x}^t = x^t/(p^t x^t)$, be its expenditure-normalized version.

Then $\mathcal{D}$ obeys GAPP if and only if $\widetilde{\mathcal{D}}$ obeys GARP.

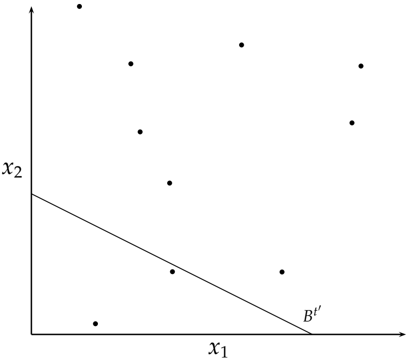
This proposition allows us to test the random expenditure augmented utility model:

it is the *same* as testing the McFadden-Richter model after suitable scaling of demand bundles.

# Observed Choices

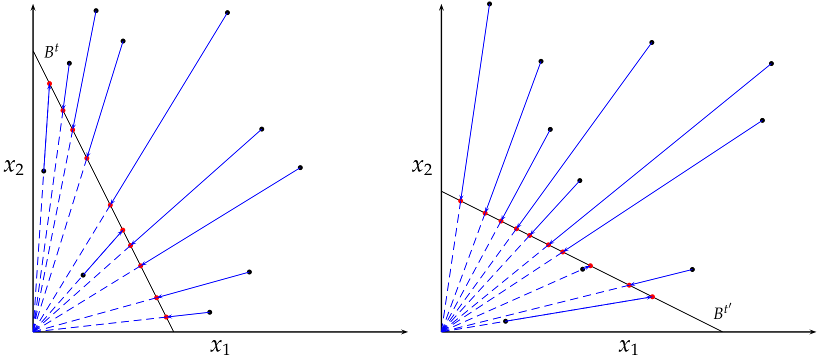



(a) $p^t = (1, 2)$



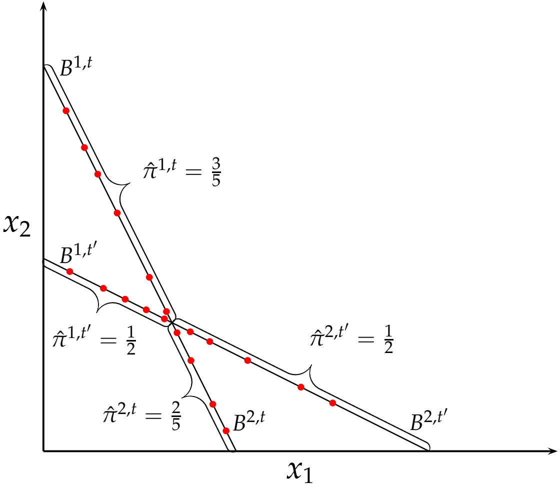

(b) $p^t = (2, 1)$

# Scaled Choices




(a) $p^t = (1, 2)$ (b) $p^t = (2, 1)$

# Scaled Empirical Choice Probabilities



# Procedure for testing the random augmented utility model

1. Fix an arbitrary expenditure, say 1.
2. At each $t$, scale each observed choice $x^t$ to $\frac{x^t}{p^t x^t}$.
3. This yields a distribution of choices on $T$ budget planes.
4. Test stochastic GARP on these scaled empirical choice probabilities.

Theorem 2: $\mathscr{D}$ can be rationalized by the random augmented utility model if and only if stochastic GARP holds on the scaled data set.

So we are using the McFadden-Richter result, but not in the way they envisaged.

# Results with Canadian Data

| Province | 97-02 | 98-03 | 99-04 | 00-05 | 01-06 | 02-07 | 03-08 | 04-09 |
|---|---|---|---|---|---|---|---|---|
| **Alberta** | **2.48** | **0** | **0** | **0** | **0** | **.003** | **.003** | **3.93** |
| p-value | .30 | 1 | 1 | 1 | 1 | 1 | 1 | .05 |
| **BC** | **2.42** | **1.50** | **1.94** | **.450** | **.436** | **5.14** | **6.32** | **7.43** |
| p-value | .19 | .23 | .71 | .62 | .64 | .02 | .02 | .03 |
| **Manitoba** | **.201** | **0** | **0** | **0** | **0** | **0** | **.027** | **.037** |
| p-value | .99 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| **New Brunswick** | **.003** | **.002** | **.002** | **0** | **0** | **1.08** | **1.11** | **1.44** |
| p-value | .99 | 1 | 1 | 1 | 1 | .31 | .49 | .26 |
| **Newfoundland** | **5.25** | **.400** | **.181** | **.216** | **.278** | **2.66** | **2.03** | **1.80** |
| p-value | .02 | .88 | .89 | .88 | .88 | .28 | .41 | .37 |
| **Nova Scotia** | **.001** | **0** | **0** | **0** | **0** | **0** | **.491** | **.590** |
| p-value | .99 | 1 | 1 | 1 | 1 | 1 | .83 | .73 |
| **Ontario** | **.061** | **.034** | **.032** | **0** | **0** | **0** | **0** | **0** |
| p-value | 1 | .98 | .99 | 1 | 1 | 1 | 1 | 1 |
| **Quebec** | **.122** | **.101** | **0** | **0** | **0** | **.991** | **1.01** | **1.29** |
| p-value | .92 | .74 | 1 | 1 | 1 | .37 | .46 | .22 |
| **Sakatchewan** | **.973** | **.636** | **.405** | **.322** | **.008** | **.009** | **.009** | **.009** |
| p-value | .60 | .47 | .44 | .46 | .99 | 1 | 1 | 1 |

# Extracting welfare bounds on prices

Pairwise comparisons of revealed price preferences for the province of Ontario, based on data set from 1999 to 2003:

| | **1999** | **2000** | **2001** | **2002** | **2003** |
|---|---|---|---|---|---|
| **1999** | | [.98,1] | [.80,.86] | [.60,.70] | [.73,.83] |
| **2000** | [0,.02] | | [.38,.46] | [.35,.45] | [.66,.71] |
| **2001** | [.14,.19] | [.55,.62] | | [.39,.45] | [.73,.80] |
| **2002** | [.30,.40] | [.55,.65] | [.55,.62] | | [.89,.93] |
| **2003** | [.17,.24] | [.24,.29] | [.20,.25] | [.07,.10] | |

The rationalizability test is based on Kitamura and Stoye (2018).

The estimates of welfare bounds require new econometric techniques.

# Conclusion

We develop a simple and intuitive model that allows for consistent welfare comparisons between prices.

This model is easily extendable to a random utility framework.

Cross sectional data sets of the type commonly collected can be used to nonparamterically test this random utility model.

We develop novel inference theory which can be used to estimate welfare bounds and for other applications.

Application on Canadian and UK household spending data:

- Model is not generally rejected.
- However, there are some rejections which implies the test has power.
- Welfare bounds are tight enough to be informative.