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AGGREGATING THE SINGLE CROSSING PROPERTY: THEORY AND APPLICATIONS TO COMPARATIVE STATICS AND BAYESIAN GAMES

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Aggregating the single crossing property: theory and applications to comparative statics and Bayesian games

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Abstract: The single crossing property plays a crucial role in monotone comparative statics (Milgrom and Shannon (1994)), yet in some important applications the property cannot be directly assumed or easily derived. Difficulties often arise because the property cannot be aggregated: the sum of two functions with the single crossing property need not have the same property. We obtain the precise conditions under which functions with the single crossing property add up to functions with this property. We apply our results to certain Bayesian games where establishing the monotonicity of strategies is an important step in proving equilibrium existence. In particular, we find conditions under which first-price auctions have monotone equilibria, generalizing the result of Reny and Zamir (2004).

Keywords: monotone comparative statics, single crossing property, Bayesian games, monotone strategies, first-price auctions, logsupermodularity.

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1. Introduction

Consider the following basic problem in comparative statics: an agent chooses the action $x \in X \subset \mathbb{R}$ to maximize her objective $v(x;s)$, where $s \in S \subset \mathbb{R}$ is some parameter; how does $\text{argmax}_{x\in X} v(x;s)$ vary with s? It is well-known that $\text{argmax}_{x\in X} V(x;s)$ increases with s if the family $\{v(\cdot;s)\}_{s\in S}$ obeys *single crossing differences*; this means that, for any $x'' > x'$, the function $\delta(s) = V(x'';s) - V(x';s)$ has the single crossing property, in the sense that δ crosses the horizontal axis just once, from negative to positive, as s increases (see Milgrom and Shannon (1994)). This simple but powerful result is useful when one is interested in comparative statics for its own sake (for example, when considering an agent's portfolio allocation problem) or when monotonicity is needed for establishing some other result (like equilibrium existence in supermodular games (see Milgrom and Roberts (1990) and Vives (1990)).

However, single crossing differences cannot always be directly assumed or easily derived from primitive assumptions. In certain problems, especially problems involving uncertainty, establishing that the property holds for a particular family of objective functions is nontrivial. For example, consider an agent who maximizes expected payoff $V(x;s) = \int_T v(x;s,t)\, dF(t)$, where t represents a possible state of the world and F the distribution over those states. Suppose that $\{v(\cdot;s,t)\}_{s\in S}$ obeys single crossing differences, so that the optimal action increases with parameter s if the state t is known; in general, this is *not* sufficient to guarantee that $\{V(\cdot;s)\}_{s\in S}$ obeys single crossing differences, so we cannot conclude that $\text{argmax}_{x\in X} V(x;s)$ increases with s.

A similar problem arises in an n-player Bayesian games where each player i takes an action after observing a signal $s_i \in [0,1]$. Signal s_i may convey direct information about player i's payoff function and also indirect information on the actions of other players in the game (through the joint distribution on players' signals). In this case, it can be shown that the player i's objective function takes the form

$$V_i(x; s_i) = \int_{[0,1]^{n-1}} v_i(x; s)\, dF(s_{-i}|s_i)\, ds_{-i}, \tag{1}$$

where $F(s_{-i}|s_i)$ gives the distribution of s_{-i} conditional on observing s_i. The existence of a Bayesian-Nash equilibrium where each player plays a monotone strategy (i.e., a strategy where the action increases with the player's signal) hinges on whether a particular player has an optimal strategy that is monotone, given that other players are playing monotone strategies (see Athey (2001)). Since, in essence, this involves checking that $\text{argmax}_{x \in X} V_i(x; s_i)$ is increasing in s_i, it is desirable to have $\{V_i(\cdot; s_i)\}_{s_i \in [0,1]}$ obey single crossing differences; however, this property may not hold, even when $\{v_i(\cdot; s)\}_{s \in [0,1]^n}$ obeys single crossing differences and the signals are affiliated.

While the problems we considered could be solved in specific contexts using various ad hoc techniques, there has been no attempt at developing a general theory that addresses them systematically. We think that these problems are best understood as arising from the fact that the single crossing property is not preserved with aggregation; in other words, the sum of two functions with the single crossing property does not generally add up to another function with this property. In this paper we provide a careful examination of the conditions under which the single crossing property *is* preserved with aggregation. Obviously, one situation in which this is true is when the functions are increasing, so that the sum is also increasing – what is interesting is that this is not the only situation in which the single crossing property is preserved. We demonstrate the value of our theory by showing how various known – and seemingly disparate – techniques for establishing the single crossing property can be understood as special cases of the theory and also show how it could lead to more general results in some significant applications.

The paper is organized as follows. In Section 2, we provide basic definitions and more careful motivation for the paper. Section 3 identifies the precise mathematical conditions under which the single crossing property is preserved by aggregation.

Those results are applied in Section 4 to the problem of a risk averse monopolist who makes output decisions under uncertainty; we identify conditions under which optimal output falls as input prices increase. While this problem may seem very basic, to the best of our knowledge, it has not been solved at this level of generality.

Sections 5 and 6 deal with the aggregation issues that arise when considering functions with the single crossing property that are defined on multi-dimensional domains. In particular, we identify the condition on $\{v_i(\cdot; s)\}_{s\in[0,1]^n}$ that is necessary and sufficient to guarantee that $\{V_i(\cdot; s_i)\}_{s_i\in[0,1]}$ (as defined by (1)) obeys single crossing differences, given that the signals $\{s_i\}_{i\geq 1}$ are affiliated. The well-known result that V_i is logsupermodular in $(x; s_i)$ when v_i is logsupermodular in (x, s) can be obtained as a special case of our result. In another application, we consider a Bayesian game where risk averse firms producing substitutable (but non-identical) goods engage in price competition. We identify conditions under which each firm's optimal strategy is monotone, given that other firms are playing monotone strategies, which (along with some other ancillary conditions) is sufficient to guarantee the existence of a Bayesian-Nash equilibrium in monotone strategies (via Athey's (2001) equilibrium existence result).

Lastly, in Section 7, we use our results to extend the work of Reny and Zamir (2004), who identified very general conditions under which a first-price (single-unit) auction has an equilibrium in monotone bidding strategies; unlike the previous literature, they allow for multiple and asymmetric bidders with interdependent values. A crucial assumption in their theorem is that the payoff difference to each bidder between making a high bid and a low bid becomes weakly smaller (in absolute terms) as the state becomes more favorable (formally if s is higher). While this assumption is reasonable in many settings, it can be violated if a higher bid imposes an opportunity cost on the bidder that is higher in more favorable states. We use our techniques to weaken their condition: loosely speaking, we show that an equilibrium exists so long as the payoff difference between a high bid and a low bid becomes smaller in relative terms as the state becomes more favorable.[1]

2. Basic Concepts and Motivation

Let $(\mathbb{S}, \geq)$ be a partially ordered set and consider a family of functions $\{u(\cdot; s)\}_{s\in\mathbb{S}}$, where each $u(\cdot; s)$ is real-valued and has domain $X \subset \mathbb{R}$. We interpret X as the possible actions of an agent, $u(\cdot; s)$ as his objective function, and s as some parameter. We say that the family $\{u(\cdot; s)\}_{s\in\mathbb{S}}$ obeys *single crossing differences* if, for any $x'' > x'$, the function $\Delta : \mathbb{S} \to \mathbb{R}$ defined by $\Delta(s) = u(x'', s) - u(x', s)$ has the single crossing property. A function Δ has the *single crossing property* if it satisfies the following:

$$\Delta(s') \geq (>)\, 0 \Longrightarrow \Delta(s'') \geq (>)\, 0 \;\text{ whenever } s'' > s'. \tag{2}$$

In the case where $\mathbb{S}$ is an interval of the real line, the graph of Δ is a curve that crosses the horizontal axis just once, hence the term 'single crossing'. We refer to a function that obeys the single crossing property as a *single crossing function* or an $\mathcal{S}$ function. Clearly, $\{u(\cdot; s)\}_{s\in\mathbb{S}}$ obeys single crossing differences if it obeys *increasing differences*; by this we mean that, for any $x'' > x'$, Δ is an increasing function.[2]

Single crossing differences is important because it guarantees that $X^*(s) \equiv \text{argmax}_{x\in\mathbb{X}} u(x; s)$ is increasing with s. Since $X^*(s)$ is not necessarily unique, we need to explain what we mean when we say that it increases with s. Let K' and K'' be two subsets of $\mathbb{R}$; K'' dominates K' in the *strong set order* (we write $K'' \geq K'$) if for any for x'' in K'' and x' in K', we have $\max\{x'', x'\} \in K''$ and $\min\{x'', x'\} \in K'$.[3] It follows immediately from this definition that if $K'' = \{x''\}$ and $K' = \{x'\}$, then $x'' \geq x'$. More generally, suppose that both sets contain their largest and smallest elements.

[1] Strictly speaking, our result does not fully generalize Reny and Zamir's because we impose an additional (but, we think, innocuous) assumption that each agent's payoff is decreasing in his bid.

[2] Our use of the term *single crossing differences* follows Milgrom (2004). In Milgrom and Shannon (1994), a family of objective functions that obey (what we call) single crossing differences is said to obey the single crossing *property*.

[3] See Topkis (1998); note that this definition of the strong set order makes sense on any lattice.

Then it is clear that the largest (smallest) element in K'' is larger than the largest (smallest) element in K'.[4] The following fundamental result says that single crossing differences is sufficient and (in a sense) necessary for monotone comparative statics.

Theorem 1 (Milgrom and Shannon (1994)) *The family* $\{u(\cdot; s)\}_{s\in\mathbb{S}}$ *obeys single crossing differences if and only if* $\text{argmax}_{x\in Y} u(x; s'') \geq \text{argmax}_{x\in Y} u(x; s')$ *for any* $s'' > s'$ *and any* $Y \subseteq X$.[5]

The usefulness of this result depends on whether there are many modeling scenarios where single crossing differences holds; indeed, there *are* many such situations and for this reason this theorem has been extensively used in the literature. However, it is not always the case that single crossing differences can be directly assumed or easily derived, and the contribution of this paper is precisely to study some of those cases. We consider two problems that serve to motivate our paper.

Problem 1. Consider an optimization problem in which the payoff of action x in state t is $v(x; s, t)$, where s is a parameter. We assume that t is drawn from the interval $T \subset \mathbb{R}$ and for each t, the family $\{v(\cdot; s, t)\}_{s\in\mathbb{S}}$ obeys single crossing differences. If the agent knows for certain that state $t = \bar{t}$ will occur, then he chooses action x to maximize $v(x; s, \bar{t})$; since $\{v(\cdot; s, \bar{t})\}_{s\in\mathbb{S}}$ obeys single crossing differences, we may conclude that the optimal action x will increase with the parameter s. However, since the state is uncertain, the agent maximizes his expected utility $V(x; s) = \int_T v(x; s, t)\, \lambda(t)\, dt$, where λ is the subjective probability of state t. If we wish to guarantee that the optimal action increases with s, we need $\{V(\cdot; s)\}_{s\in\mathbb{S}}$ to obey single crossing differences.

[4]Throughout this paper, when we say that something is 'greater' or 'increasing', we mean to say that it is greater or increasing in the *weak* sense. Most of the comparisons in this paper are weak, so this convention is convenient. When we are making a strict comparison, we shall say so explicitly, as in 'strictly higher', 'strictly increasing', etc.

[5]This is a special case of Milgrom and Shannon's result, which considers the more general case where X is a lattice rather than a subset of $\mathbb{R}$.

In other words, for any $x'' > x'$, we require the map $s \mapsto V(x''; s) - V(x'; s)$ to be an $\mathcal{S}$ function. Note that

$$\begin{aligned} V(x''; s) - V(x'; s) &= \int_T [v(x''; s, t) - v(x', s, t)]\ \lambda(t)\, dt \\ &= \int_T \Delta(s, t)\lambda(t)\, dt, \end{aligned}$$

where, for each t, the function $\Delta(\cdot; t)$ given by $\Delta(s; t) = v(x''; s, t) - v(x', s, t)$ is an $\mathcal{S}$ function (since $\{v(\cdot; s, \bar{t})\}_{s\in\mathbb{S}}$ obeys single crossing differences). So effectively we face the following problem: *when is an integral (or sum) of single crossing functions a single crossing function?*

Problem 2. Let $\{u(\cdot; s)\}_{s\in\mathbb{S}}$ be a family of functions parameterized by s, where, in this case, we assume that $\mathbb{S}$ is an interval of $\mathbb{R}$. We assume that this family obeys single crossing differences, so that the optimal action is increasing in s. Interpreting s to be the state of the world, an agent has to choose his action x under uncertainty (before s is realized) but after observing a signal $\theta \in \Theta \subset \mathbb{R}$. The agent maximizes the expected utility $U(x; \theta) = \int_{\mathbb{S}} u(x; s)\, \lambda(s|\theta)\, ds$ where $\lambda(\cdot|\theta)$ is his posterior distribution over s after observing signal θ. It is natural to expect, given our assumptions, that the optimal action will be higher if higher states are more likely. Indeed, it is possible to formalize this intuition. Suppose that the joint distribution over (s, θ) is *affiliated* in the sense that its distribution is represented by a logsupermodular density function. In this case, the posterior density functions $\{\lambda(\cdot|\theta)\}_{\theta\in\Theta}$ are ordered by the *monotone likelihood ratio* (MLR), i.e., whenever $\theta'' > \theta'$, the ratio $\lambda(s|\theta'')/\lambda(s|\theta')$ is increasing in s. This guarantees that the family $\{U(\cdot; \theta)\}_{\theta\in\Theta}$ obeys single crossing differences (see Athey (2002)), so that higher signals lead to higher actions.

However, in many problems, the state cannot be adequately captured by a one-dimensional variable. So consider instead the case where the state $s = (s_1, s_2) \in \mathbb{S}_1 \times \mathbb{S}_2$, where $\mathbb{S}_i$ (for $i = 1$ and 2) are intervals of $\mathbb{R}$. Once again assume that $\{u(\cdot; s)\}_{s\in\mathbb{S}}$ obeys single crossing differences and that the joint density function λ over (s_1, s_2, θ) is logsupermodular. Is this sufficient to guarantee that $\{U(\cdot; \theta)\}_{\theta\in\Theta}$

obeys single crossing differences? It turns out that the answer is 'No'; to see why that may be so, let x'' and x' be two actions. Then

$$\begin{aligned} U(x'';\theta) - U(x';\theta) &= \int_{S_2}\int_{S_1} [u(x'';s_1,s_2) - u(x';s_1,s_2)]\ \lambda(s_1,s_2|\theta)\,ds_1\,ds_2 \\ &= \int_{S_2} \Delta(\theta;s_2)ds_2 \end{aligned}$$

where $\Delta(\theta;s_2) = \int_{S_1} [u(x'';s_1,s_2) - u(x';s_1,s_2)]\ \lambda(s_1,s_2|\theta)\,ds_1$. It is not hard to see that the one-dimensional result cited above allows us to conclude that, for *each* s_2, $\Delta(\cdot;s_2)$ is an $\mathcal{S}$ function (of θ). However, to guarantee that $\{U(\cdot;\theta)\}_{\theta\in\Theta}$ obeys single crossing difference (equivalently, to guarantee that the map $\theta \mapsto U(x'';\theta) - U(x';\theta)$ is an $\mathcal{S}$ function), we face, once again, the following question: *when is an integral (or sum) of single crossing functions a single crossing function?*

The next section provides a systematic treatment of this issue.

3. Aggregating Single Crossing Functions

It is straightforward to check that the sum of two single crossing (or $\mathcal{S}$) functions is not necessarily a single crossing function. However, the property *is* preserved by aggregation if the two $\mathcal{S}$ functions are related in a particular way. This section is devoted to identifying that relation and examining its properties.

Consider the set of $\mathcal{S}$ functions defined on the partially ordered set $(\mathbb{S},\geq)$. We define the binary relation $\sim$ on this set in the following way. We say that $h \sim g$ if (a) at any $s' \in \mathcal{S}$, such that $g(s') < 0$ and $h(s') > 0$, we have

$$-\frac{g(s')}{h(s')} \geq -\frac{g(s'')}{h(s'')} \text{ when } s'' > s';\ \textit{and} \tag{3}$$

(b) at any $s' \in \mathcal{S}$, such that $h(s') < 0$ and $g(s') > 0$, we have

$$-\frac{h(s')}{g(s')} \geq -\frac{h(s'')}{g(s'')} \text{ when } s'' > s'.$$

It follows immediately from this definition that $\sim$ is a reflexive relation and it is easy to see that it is not transitive. A quick check will also confirm that if $h \sim g$ then

$\alpha h \sim \beta g$ where α and β are nonnegative scalars. We shall refer to two single crossing functions that are related by $\sim$ as $\mathcal{S}$-*summable*; a family of single crossing functions in which any two functions are related to each other is said to be an $\mathcal{S}$-summable family. The motivation for this term and the significance of the relation $\sim$ rests on the following result.

Proposition 1 *Let h and g be two $\mathcal{S}$ functions. Then $\alpha h + g$ is an $\mathcal{S}$ function for all positive scalars α if and only if $h \sim g$.*

Proof: We first show that it is necessary for $h \sim g$. With no loss of generality, suppose $g(s') < 0$ and $h(s') > 0$. Define $\alpha' = -g(s')/h(s') > 0$ and note that $\alpha' h(s') + g(s') = 0$. Since $\alpha' h + g$ is an $\mathcal{S}$ function, for any $s'' > s'$, we have $\alpha' h(s'') + g(s'') \geq 0$. Re-arranging this inequality and bearing in mind that $h(s'') > 0$ (since h is an $\mathcal{S}$ function and $h(s') > 0$), we obtain

$$\alpha' = -\frac{g(s')}{h(s')} \geq -\frac{g(s'')}{h(s'')}.$$

For the other direction, suppose

$$\alpha h(s') + g(s') \geq (>)\, 0. \tag{4}$$

If $g(s') \geq 0$ and $h(s') \geq 0$ then we have $g(s'') \geq 0$ and $h(s'') \geq 0$ since g and h are $\mathcal{S}$ functions. It follows that

$$\alpha h(s'') + g(s'') \geq 0. \tag{5}$$

If the inequality (4) is strict then (5) will also be strict since either $g(s') > 0$ or $h(s') > 0$.

Now consider the case where (4) holds but one of the two functions is negative at s'. Suppose that $g(s') < 0$. Then $h(s') > 0$ since (4) holds. For any $s'' > s'$,

$$\alpha \geq (>) - \frac{g(s')}{h(s')} \geq -\frac{g(s'')}{h(s'')}$$

where the first inequality follows from (4) and the second from the fact that $f \sim g$. Re-arranging this inequality, we obtain $\alpha h(s'') + g(s'') \geq (>)\, 0$. (Note that $h(s'') > 0$

since h is an $\mathcal{S}$ function and $h(s') > 0$.) **QED**

It is clear that if h and g are increasing functions then they are $\mathcal{S}$-summable. For another simple example, suppose $\mathbb{S} = \mathbb{R}$ and consider $h(s) = s^2 + 1$ and $g(s) = s^3$. In this case h is not an increasing function but we still have $h \sim g$. This is easy to check: for $s > 0$, we have $-g(s)/h(s) < 0$, while for $s < 0$, the ratio $-g(s)/h(s) = -s^3/(s^2+1)$ is positive and decreasing in s. Therefore, by Proposition 1, $f(s) = \alpha(s^2 + 1) + s^3$ is an $\mathcal{S}$ function for any $\alpha > 0$. Note that if h and g are $\mathcal{S}$-summable functions defined on a subset of $\mathbb{R}$, then $\tilde{h} = h \circ \phi$ and $\tilde{g} = g \circ \phi$ are also $\mathcal{S}$-summable, where ϕ is an increasing function defined on a (not necessarily one-dimensional) domain. For example, if we choose $\phi(s_1, s_2) = s_1 + s_2$, the functions $\tilde{h}(s_1, s_2) = (s_1 + s_2)^2 + 1$ and $\tilde{g}(s_1, s_2) = (s_1 + s_2)^3$ are $\mathcal{S}$-summable on the domain $\mathbb{R} \times \mathbb{R}$.

The next result is a natural extension of Proposition 1.

Proposition 2 *Suppose $\mathcal{F} = \{f_i\}_{1 \leq i \leq M}$ is an $\mathcal{S}$-summable family. (i) Then $\sum_{i=1}^{M} \alpha_i f_i$, where $\alpha_i \geq 0$ for all i, is an $\mathcal{S}$ function. (ii) Suppose h is an $\mathcal{S}$ function and $h \sim f_i$ for all i. Then $h \sim \sum_{i=1}^{M} f_i$.*

Proof: (i) Note that if $\mathcal{F}$ is an $\mathcal{S}$-summable family then $\{\alpha_i f_i\}_{1 \leq i \leq M}$ (where $\alpha_i \geq 0$ for $i = 1, 2, ..., M$) is also an $\mathcal{S}$-summable family. Given this, we need only show that $F = \sum_{i=1}^{M} f_i$ is an $\mathcal{S}$ function.

Suppose that $F(s') \geq 0$; we are required to show that $F(s'') \geq 0$ for any $s'' > s'$. If $f_i(s') \geq 0$ for all i, then $f_i(s'') \geq 0$ for all i, so we obtain $F(s'') \geq 0$. Consider next the case where $f_i(s') < 0$ for some i. In this case, we may partition $\mathcal{F}$ into three subsets; for $f_i \in \mathcal{F}_1$, we have $f_i(s') < 0$; for $f_i \in \mathcal{F}_2$, we have $f_i(s') > 0$; and for $f_i \in \mathcal{F}_3$, we have $f_i(s') = 0$. Since $\mathcal{F}_1$ is nonempty, so is $\mathcal{F}_2$. By 'splitting' the functions in $\mathcal{F}_1$ and $\mathcal{F}_2$ if necessary and adding functions in $\mathcal{F}_1$ with those in $\mathcal{F}_2$, we may write $F = \sum_{j=1}^{L} g_j$ such that $g_j(s') \geq 0$ for all j and each g_j is the weighted sum of *at* *most two* functions in $\mathcal{F}$; formally, for each g_j there are functions f_m and f_n in $\mathcal{F}$ and nonnegative scalars β_m and β_n such that $g_j = \beta_m f_m + \beta_n f_n$. By Proposition 1, g_j is an $\mathcal{S}$ function, so we have $g_j(s'') \geq 0$ for all j. This gives $F(s'') \geq 0$. A straightforward modification of this argument shows that if $F(s') > 0$ then $F(s'') > 0$.

(ii) In this case, $\{f_1, f_2, ..., f_M, h\}$ form an $\mathcal{S}$-summable family so, by (i), $\sum_{i=1}^M f_i$ is an $\mathcal{S}$ function and, for any $\alpha > 0$, the function $\alpha h + \left(\sum_{i=1}^M f_i\right)$ is also an $\mathcal{S}$ function. By Proposition 1, this implies that $h \sim \sum_{i=1}^M f_i$. QED

As an illustration of Proposition 2, note that $f_1(s) = 1/s$, $f_2(s) = -1/s^2$ and $f_3(s) = 1$ (defined on $\mathbb{S} = \mathbb{R}_+$) are $\mathcal{S}$-summable; hence each of these functions is $\sim$-related to the $\mathcal{S}$ function $\psi(s) = s^{-1} - 2s^{-2} + 1$ (amongst others). The curve of ψ crosses the horizontal axis once at $s = 1$, has its peak at $(4, 1.125)$, and approaches 1 as s approaches infinity. In the light of Proposition 2, the next result should not be surprising. It says that the integral of an $\mathcal{S}$-summable family is an $\mathcal{S}$ function.

Theorem 2 *Let T be a measurable subset of $\mathbb{R}$ and $\{f(\cdot, t)\}_{t \in T}$ an $\mathcal{S}$-summable family indexed by elements in T and defined on $\mathbb{S}$. For any fixed s, $f(s, \cdot)$ is a measurable and bounded function (of t). (i) Then the function $F : \mathbb{S} \to \mathbb{R}$ defined by $F(s) = \int_T f(s,t)dt$ is also an $\mathcal{S}$ function. (ii) If g is an $\mathcal{S}$ function and $g \sim f(\cdot, t)$ for all $t \in T$, then $g \sim F$.*

The proof of Theorem 2 is in the Appendix. The next result is, in one guise or another, well-known, and has many applications, especially in comparative statics problems under uncertainty (see, for example, Jewitt (1987), Gollier (2001), and Athey (2002)). It is a straightforward consequence of Theorem 2 (for the alternative and more familiar proof see, for example, Athey (2002)).

Corollary 1 *Let T be a measurable subset of $\mathbb{R}$ and K a subset of $\mathbb{R}$. Suppose $f : T \to \mathbb{R}$ is an $\mathcal{S}$ function and that $g : T \times K \to \mathbb{R}_{++}$ is a logsupermodular* *function.*[6] *Then* $F : K \to \mathbb{R}$ *is an* $\mathcal{S}$ *function, where* $F(k) = \int_T f(t)g(t,k)\,dt$.

Proof: By Theorem 2, we need only show that $\{\phi(\cdot,t)\}_{t\in T}$, where $\phi(k,t) = f(t)g(t,k)$, is an $\mathcal{S}$-summable family (parameterized by $t \in T$). Note that $\phi(\cdot,t)$ is either a positive or negative (depending on the sign of $f(t)$), so it is clearly an $\mathcal{S}$ function (of k). Suppose $\phi(k^*,t') < 0$ and $\phi(k^*,t'') > 0$. Since f is an $\mathcal{S}$ function, $t'' > t'$. For $k^{**} > k^*$ we have

$$\begin{aligned} -\frac{\phi(k^*,t')}{\phi(k^*,t'')} &= -\frac{f(t')g(t',k^*)}{f(t'')g(t'',k^*)} \geq -\frac{f(t')g(t',k^{**})}{f(t'')g(t'',k^{**})} \\ &= -\frac{\phi(k^{**},t')}{\phi(k^{**},t'')}, \end{aligned}$$

where the inequality follows from the logsupermodularity of g. So we have shown that $\phi(\cdot,t') \sim \phi(\cdot,t'')$. **QED**

Suppose $\mathbb{S} = [\underline{s},\bar{s}]$ and let f be a bounded and measurable function defined on this interval. For some point $\hat{s}$ in the interior of $\mathbb{S}$, and $a \leq \hat{s}$, define the function $\hat{f} : \{a\} \cup (\hat{s},\bar{s}] \to \mathbb{R}$ by

$$\hat{f}(s) = \begin{cases} \int_{[\underline{s},\hat{s}]} f(z)dz & \text{if } s = a \\ f(s) & \text{if } s \in (\hat{s},\bar{s}] \end{cases} \tag{6}$$

It is clear that this *domain coarsening* preserves the single crossing property, in the sense that $\hat{f}$ is an $\mathcal{S}$ function if f is an $\mathcal{S}$ function. Indeed, we may go further. The function $\bar{f}$, defined on the two-point domain $\{0,1\}$ by

$$\hat{f}(s) = \begin{cases} \int_{[\underline{s},\hat{s}]} f(z)dz & \text{if } s = 0 \\ \int_{(\hat{s},\bar{s}]} f(z)dz & \text{if } s = 1 \end{cases} \tag{7}$$

is also an $\mathcal{S}$ function if f is an $\mathcal{S}$ function. The next result states that the $\sim$ relation is closed under domain coarsening; it will be extended in Section 5 and is eventually used in our proof of equilibrium existence in first-price auctions in Section 7.

[6]We define a function as *logsupermodular* if its logarithm is a supermodular function.

Proposition 3 *Suppose f and g are bounded and measurable $\mathcal{S}$ functions defined on $\mathbb{S} = [\underline{s}, \bar{s}]$. (i) Then the functions $\hat{f}$ and $\hat{g}$ (with $\hat{g}$ defined by (6), in an analogous way to $\hat{f}$) satisfy $\hat{f} \sim \hat{g}$. (ii) The functions $\bar{f}$ and $\bar{g}$ (with $\bar{g}$ defined by (7), in an analogous way to $\bar{f}$) also satisfy $\bar{g} \sim \bar{f}$.*

Proof: We omit the proof of (ii), which is similar to that of (i). To prove (i), note that (by Proposition 1) it suffices to show that $h = \alpha\hat{f} + \hat{g}$ is an $\mathcal{S}$ function for any positive scalar α. The only interesting case to consider is the one where $h(a) = \alpha\hat{f}(a) + \hat{g}(a) \geq (>)\, 0$. Then $\int_{[\underline{s},\hat{s}]} \alpha f(z) + g(z)\, dz \geq (>)\, 0$, which implies that there is $\tilde{z} \in [\underline{s}, \hat{s}]$ such that $\alpha f(\tilde{z}) + g(\tilde{z}) \geq (>)\, 0$. This guarantees that $h(s) = \alpha f(s) + g(s) \geq (>)\, 0$ for all $s > \hat{s} \geq \tilde{z}$, since $\alpha f + g$ is an $\mathcal{S}$ function (by Proposition 1 again). QED

4. Single crossing differences and comparative statics

We are now in a position to re-visit *Problem 1* (see Section 2), in which an agent chooses x to maximize $V(x; s) = \int_T v(x; s, t)\, \lambda(t)\, dt$. To guarantee that the optimal choice of x increases with the parameter s, it suffices that $\{V(\cdot; s)\}_{s\in\mathbb{S}}$ obeys single crossing differences; Theorem 2 tells us that this holds if, for any $x'' > x'$ and t, the function $\Delta(\cdot, t)$ given by

$$\Delta(s,t) = v(x''; s, t) - v(x', s, t), \tag{8}$$

is an $\mathcal{S}$ function (of s) and that $\{\Delta(\cdot, t)\}_{t\in T}$ is an $\mathcal{S}$-summable family.

For this result to be useful it is important that there is a simple way of checking that a family of single crossing functions is $\mathcal{S}$-summable. In many applications, $v = h \circ \phi$, where ϕ is the (monetary) payoff, which depends on (x, s, t), and h is the agent's Bernoulli utility function. In this case, it is possible to write down conditions on ϕ and h which together guarantee that $\{\Delta(\cdot, t)\}_{t\in T}$ form an $\mathcal{S}$-summable family. We provide this in the next proposition.

Proposition 4 *Let* X *be a subset of* $\mathbb{R}$*,* $\mathbb{S}$ *a partially ordered set, and* T *a totally ordered set. Then* $\{\Delta(\cdot, t)\}_{t \in T}$ *(with* Δ *defined by (8) and where* $v = h \circ \phi$*) is an* $\mathcal{S}$*-summable family if* $\phi : X \times \mathbb{S} \times T \to \mathbb{R}$ *and* h *have the following properties:*
(i) for any given t*,* $\{\phi(\cdot; s, t)\}_{s \in \mathbb{S}}$ *obeys increasing differences;*
(ii) for any given s*,* $\{\phi(\cdot; s, t)\}_{t \in T}$ *obeys single crossing differences;*
(iii) $h : \mathbb{R} \to \mathbb{R}$ *is twice differentiable, with* $h' > 0$ *and* $-h''(z)/h'(z)$ *decreasing in* z*, i.e.,* h *exhibits decreasing absolute risk aversion (DARA);*
(iv) ϕ *is increasing in* (s, t)*; and*
(v) any *of the following: (a) for every* x *and* $s'' > s'$*,* $\phi(x, s'', t) - \phi(x, s', t)$ *is independent of* t*; (b)* h *is concave and for every* x *and* $s'' > s'$*,* $\phi(x, s'', t) - \phi(x, s', t)$ *decreases with* t*; or (c)* h *is convex and for every* x *and* $s'' > s'$*,* $\phi(x, s'', t) - \phi(x, s', t)$ *increases with* t*.*

The proof of Proposition 4 is in the Appendix. Clearly, condition (i) implies that Δ is a single crossing function of s, so the non-trivial part of this result consists of showing that all the conditions together are sufficient to guarantee that $\{\Delta(\cdot, t)\}_{t \in T}$ form an $\mathcal{S}$-summable family. Note that in the case where S and T are intervals and ϕ is differentiable, the conditions on ϕ are simple to express: (i) is equivalent to $\partial^2 \phi / \partial x \partial s \geq 0$; a sufficient (though not necessary) condition for (ii) is $\partial^2 \phi / \partial x \partial t \geq 0$; (iv) is equivalent to $\partial \phi / \partial s \geq 0$ and $\partial \phi / \partial t \geq 0$ and (v)-a, (v)-b, and (v)-c are equivalent to $\partial^2 \phi / \partial s \partial t$ being equal, greater than, and smaller than 0 respectively.

We shall appeal to Proposition 4 in a number of applications, beginning with the following comparative statics problem, which is seemingly basic but, as far as we know, has not been solved with the level of generality permitted here.

Application 1: Comparative statics of a monopolist under uncertainty

Consider a monopolist that has to decide on its optimal output level $x > 0$. Its profit function is $\Pi(x; s) = xP(x) - C(x; s)$, where P is the inverse demand function and $C(\cdot; s)$ is the cost function, parameterized by s in $(\mathbb{S}, \geq)$. It is well-known that a *decrease in marginal cost leads to a rise in the profit-maximizing output.* To model this formally, assume that the family $\{C(\cdot; s)\}_{s\in\mathbb{S}}$ obeys decreasing differences; if C is differentiable, this is equivalent to marginal cost dC/dx decreasing with s. It follows that $\{\Pi(\cdot; s)\}_{s\in\mathbb{S}}$ obeys increasing differences, so an application of Theorem 1 guarantees that the profit-maximizing output increases with s.

Now consider a more general setting where the firm faces uncertainty over the demand for its output. We assume that the profit at state $t \in T \subset \mathbb{R}$ is given by

$$\Pi(x; s, t) = xP(x; t) - C(x; s) \tag{9}$$

and that the firm maximizes $V(x; s) = \int_T h\left(\Pi(x; s, t)\right) \lambda(t)\, dt$, where $\lambda(t)$ is the subjective probability of state t and $h : \mathbb{R} \to \mathbb{R}$ is the Bernoulli utility function representing the monopolist's attitude towards uncertainty. We would like to identify conditions under which $\{V(\cdot; s)\}_{s\in\mathbb{S}}$ obeys single crossing differences, so that we could guarantee that the optimal output level increases with s. For each t, the family $\{\Pi(\cdot; s)\}_{s\in\mathbb{S}}$ obeys increasing differences if $\{C(\cdot; s)\}_{s\in\mathbb{S}}$ obeys decreasing differences; it follows that $\{v(\cdot; s, t)\}_{s\in\mathbb{S}}$, where $v(\cdot; s, t) = h(\Pi(\cdot; s, t))$, will obey single crossing differences. To apply Theorem 2 to this problem we need to show that $\{\Delta(\cdot, t)\}_{t\in T}$ is an $\mathcal{S}$-summable family, where

$$\Delta(s, t) = h(\Pi(x''; s, t)) - h(\Pi(x', s, t)). \tag{10}$$

The next result uses Proposition 4 to identify conditions under which this holds.

Proposition 5 *Suppose that $\{\Pi(\cdot; s, t)\}_{(s,t)\in\mathbb{S}\times T}$ is given by (9) and has the following properties: (i) C is increasing x, decreasing in s, and $\{C(\cdot; s)\}_{s\in\mathbb{S}}$ obeys decreasing differences, and (ii) P is decreasing in x and increasing in t and $\{\ln P(\cdot; t)\}_{t\in T}$ obeys increasing differences. In addition, (iii) suppose that $h : \mathbb{R} \to \mathbb{R}$ is twice differentiable, with $h' > 0$ and obeys DARA.*
Then $\{\Delta(\cdot; t)\}_{t\in T}$ (defined by (10)) is an $\mathcal{S}$-summable family, $\{V(\cdot; s)\}_{s\in\mathbb{S}}$ obeys single crossing differences, and $\operatorname{argmax}_{x\in X} V(x; s)$ increases with s.

Remark on condition (ii): Assuming that P is differentiable, $\{\ln P(\cdot;t)\}_{t\in T}$ obeys increasing differences if and only if for all $x > 0$ and $t'' > t'$,

$$-P(x;t'')\frac{dx}{dP}(x;t'') \leq -P(x;t')\frac{dx}{dP}(x;t').$$

In other words, condition (ii) says that in a high state, the market clearing price is high and the price elasticity of demand is low. Note also that when P is decreasing x and increasing in t, the assumption that $\{P(\cdot;t)\}_{t\in T}$ obeys increasing differences is stronger than the assumption that $\{\ln P(\cdot;t)\}_{t\in T}$ obeys increasing differences.

Remark on condition (iii): Like condition (iii) in Proposition 4), this does *not* require h to be concave, i.e., the firm need not be risk averse and even if it were, the firm need not face a concave maximization problem because Π need not be concave in x.

Proof: We need to check that Π and h satisfy the conditions of Proposition 4. It is clear that with our assumptions, Π is increasing in (s,t) and that for any t, the family $\{\Pi(\cdot;s,t)\}_{s\in\mathbb{S}}$ obeys increasing differences (since $\{C(\cdot;s)\}_{s\in\mathbb{S}}$ obeys decreasing differences). Furthermore, for every x and $s'' > s'$, $\Pi(x;s'',t) - \Pi(x;s',t) = C(x;s') - C(x;s'')$, which is independent of t (so version (a) of condition (v) in Proposition 4 is satisfied). It remains for us to show that, for any given s, the family $\{\Pi(\cdot;s,t)\}_{t\in T}$ obeys single crossing differences.[7] Suppose $x'' > x'$ and $\Pi(x'';s,t') - \Pi(x';s,t') \geq (>)\ 0$. Then $x''P(x'';t') - x'P(x';t') \geq\ (>)\ 0$ since C is increasing in x.

$$\begin{aligned} x''P(x'';t') - x'P(x';t') &= \left[x''\frac{P(x'';t')}{P(x';t')} - x'\right]P(x';t') \\ &\leq \left[x''\frac{P(x'';t'')}{P(x';t'')} - x'\right]P(x';t') \\ &\leq \left[x''\frac{P(x'';t')}{P(x';t'')} - x'\right]P(x';t'') \\ &= x''P(x'';t'') - x'P(x';t'') \end{aligned}$$

[7] The argument we use to show that $\{\Pi(\cdot;s,t)\}_{t\in T}$ obeys single crossing differences is an adaptation of the one used by Amir (1996) to guarantee that reaction curves in the Cournot model are downward sloping.

The first inequality is true since $\{\ln P(\cdot;t)\}_{t\in T}$ obeys increasing differences. The second inequality holds since P is increasing in t and the term in the square bracket is nonnegative. We conclude that $\Pi(x''; s, t'') - \Pi(x'; s, t'') \geq\ (>)\ 0$. **QED**

Proposition 5 says that under mild assumptions, *a firm that faces lower marginal costs will increase output, even if it is operating under uncertainty.* We may understand this result in the following way. For any given s, the family $\{\Pi(\cdot; s, t)\}_{t\in T}$ obeys single crossing differences, which means that if output x'' yields higher expected utility than a lower output x', then it must be the case that x'' gives higher profit (and thus utility) than x' in the high states, and (possibly) lower profit than x' in the low states. Furthermore, a high state leads to high profit at any output level. If x'' gives higher utility than x' in a state $\bar{t}$, then it will continue to give higher utility after the fall in marginal cost. However, the *size* of that gain will typically change and will depend on the curvature of h; formally, the gain varies from $h(\Pi(x''; s^*, t)) - h(\Pi(x'; s^*, t))$ to $h(\Pi(x''; s^{**}, t)) - h(\Pi(x'; s^{**}, t))$. While the utility gains and losses (of x'' over x') at different states will change with the fall in marginal cost, the gains (which occur at higher states and thus higher profit levels) will continue to outweigh the losses provided h obeys DARA.

Special cases of Proposition 5 are known. Sandmo (1971) considered the behavior of a price-taking firm under uncertainty, with the market price experiencing additive shocks; in our notation, he assumed that $P(x;t) = \bar{P} + t$. The problem of changing marginal cost was not considered in his paper, but he showed that an increase in $\bar{P}$ leads to higher output if h obeys DARA (in other words, the firm has an upward sloping supply curve). This result is a special case of ours since there is no formal difference between a rise in the price of the good by (say) q and fall in its marginal cost by q. Note also that, unlike Sandmo's result, we do not require the concavity of the optimization problem.

Milgrom (1994) did not specifically examine the question we posed but pointed out that a large class of seemingly distinct comparative statics problems has the same solution because they all rely on the same Spence-Mirrlees condition. A special case of our problem can indeed be solved by appealing to the Spence-Mirrlees condition and checking this condition in turn provides another application of Theorem 2. To see this, suppose $P(x,t) = \bar{P}(x) + t$ (compared to Sandmo's assumption, $\bar{P}$ may now depend on x but the shock remains additive); by Theorem 1 in Milgrom (1994), $\text{argmax}_{x \in X} V(x;s)$ increases s if W_x/W_y is increasing in s, where $W(x,y,s) = \int h(y + tx - C(x,s))\lambda(t)dt$. We claim that this is guaranteed by the assumptions of Proposition 5.

Suppose, $W_x(x,y;s^*)/W_y(x,y;s^*) = \alpha$ and consider the function F defined by

$$F(s) \equiv \int h'(y + tx - C(x;s))[t - C_x(x;s) - \alpha]\lambda(t)ds. \tag{11}$$

Note that $F(s^*) = 0$. If F is an $\mathcal{S}$ function, then for $s^{**} > s^*$, we obtain $F(s^{**}) \geq 0$, which may be re-written as $W_x(x,y;s^{**})/W_y(x,y;s^{**}) \geq \alpha$, as required by Milgrom's theorem. It remains for us to show that F is an $\mathcal{S}$ function. Denoting the integrand in (11) by $f(s,t)$, it is clear that $f(\cdot,t)$ is an $\mathcal{S}$ function since marginal cost, $C_x(x;s)$, is decreasing in s. By Theorem 2, F is an $\mathcal{S}$ function if $\{f(\cdot,t)\}_{t \in T}$ forms an $\mathcal{S}$-summable family. By directly checking (3), it is not hard to show that this is true if h obeys DARA and C is increasing in s (we leave the details to the reader).

5. Integrable Single Crossing Property

In this section, we consider functions defined on the domain $\mathbb{S}$, where $\mathbb{S} = \Pi_{i=1}^n \mathbb{S}_i$, with $\mathbb{S}_i$ a bounded and measurable subset of $\mathbb{R}$. We endow $\mathbb{S}$ with the product order. For any $s \in \mathbb{S}$, we denote its subvector consisting of entries in $K \subset N = \{1, 2, ..., n\}$ by s_K and write s as $(s_{N \setminus K}, s_K)$. The set consisting of the subvectors s_K we denote by $\mathbb{S}_K$, so $\mathbb{S} = \mathbb{S}_{N \setminus K} \times \mathbb{S}_K$. For a function f defined on $\mathbb{S}$, we denote its restriction to the subvector $s_{N \setminus K}$, with s_K held fixed at s'_K, by $f(\cdot, s'_K)$. A subset of N that appears often in our exposition is $N \setminus \{k\}$; we denote this subset by N_k.

We are interested in finding conditions on $f : \mathbb{S} \to \mathbb{R}$ which guarantee that $F : \mathbb{S}_1 \to \mathbb{R}$, defined by

$$F(s_1) = \int_{\mathbb{S}_2} \int_{\mathbb{S}_3} \ldots \int_{\mathbb{S}_n} f(s_1, s_2, ..., s_{n-1}, s_n)\, ds_2\, ds_3 \ldots ds_n\,, \tag{12}$$

is an $\mathcal{S}$ function. Notice that we have already found a solution to this problem in the case where the domain of f is $\mathbb{S}_1 \times \mathbb{S}_2$: if f is an $\mathcal{S}$ function, then so is its restriction $f(\cdot, s_2)$ (for any s_2); provided the family $\{f(\cdot, s_2)\}_{s_2 \in \mathbb{S}_2}$ is $\mathcal{S}$-summable, we know from Proposition 2 that (subject to some measurability conditions) the map from s_1 to $\int_{\mathbb{S}_2} f(s_1, s_2)\, ds_2$ is also an $\mathcal{S}$ function.

More generally, Proposition 2 tells us that if $\{f(\cdot, s_n)\}_{s_n \in \mathbb{S}_n}$ is an $\mathcal{S}$-summable family, then

$$F_n(s_{N_n}) = \int_{\mathbb{S}_n} f(s_{N_n}, s_n)\, ds_n \tag{13}$$

is an $\mathcal{S}$ function. However, since $\{F_n(\cdot, s_{n-1})\}_{s_{n-1} \in \mathbb{S}_{n-1}}$ need not be an $\mathcal{S}$-summable family, the integral of F_n with respect to s_{n-1} need not be an $\mathcal{S}$ function and, by extension, neither can we guarantee that F is an $\mathcal{S}$ function. For this property to hold, we need conditions on f that guarantee the preservation of the $\mathcal{S}$-summable relation after each round of integration.

A function $f : \mathbb{S} \to \mathbb{R}$ has the *j-integrable single crossing property* if it has the single crossing property and

$$f(\cdot, s''_K) \sim f(\cdot, s'_K) \tag{14}$$

whenever $s''_K > s'_K$, for every $K \subset N_j$. We refer to such a function as an $\mathcal{I}_j$ function. Since f has the single crossing property, if $f(s^*_{N\backslash K}, s''_K)$ and $f(s^*_{N\backslash K}, s'_K)$ have opposite signs then it must be the case that the former is positive and the latter negative. Therefore, the condition (14) is equivalent to checking that (see (3))

$$-\frac{f(s^*_{N\backslash K}, s'_K)}{f(s^*_{N\backslash K}, s''_K)} \geq -\frac{f(s^{**}_{N\backslash K}, s'_K)}{f(s^{**}_{N\backslash K}, s''_K)} \text{ whenever } s^{**}_{N\backslash K} > s^*_{N\backslash K}. \tag{15}$$

If f is an $\mathcal{I}_j$ function for every $j \in N$, then we shall refer to it as an $\mathcal{I}$ function. It is certainly possible for a function to be an $\mathcal{I}_j$ function for some j without it being an $\mathcal{I}$

function. For example, suppose $\mathbb{S} = \{(1,1),(1,2),(2,1),(2,2)\}$ and let $f(1,1) = -1$, $f(2,1) = -2$, $f(1,2) = 2$, and $f(2,2) = 1$. Then f is trivially an $\mathcal{I}_2$ function, but it is not an $\mathcal{I}_1$ function because

$$-\frac{f(1,1)}{f(1,2)} = \frac{1}{2} < -\frac{f(2,1)}{f(2,2)} = 2.$$

It is straightforward to check that the function h given by $h(s) = f(s)g(s)$ is an $\mathcal{I}_j$ function if f is an $\mathcal{I}_j$ function and g is logsupermodular. In particular, any increasing function is an $\mathcal{I}$ function, so if f is increasing and g is logsupermodular, then h is an $\mathcal{I}$ function.

A useful feature of the logsupermodular property is that it holds if and only if it holds "one dimension at a time", i.e., a function g is logsupermodular if and only if $g(\cdot, s''_K)/g(\cdot, s'_K)$ is increasing in the scalar $s_{N\setminus K}$ whenever $s''_K > s'_K$ where K has exactly $n-1$ elements. The next result, which we prove in the Appendix, shows that the same is true of the $\mathcal{I}_j$ property.

Proposition 6 *Let $f : \mathbb{S} \to \mathbb{R}$ be an $\mathcal{S}$ function. Then f is an $\mathcal{I}_1$ function if and only if the following holds: $f(\cdot; s''_K) \sim f(\cdot; s'_K)$ whenever $s''_K > s'_K$, where K is a subset of N with exactly $n-1$ elements and $s'_1 = s''_1$ if $1 \in K$.*

The main result of this section is the following theorem; note that part (ii) clearly follows from (i).

Theorem 3 *Let $f : \mathbb{S} \to \mathbb{R}$ be a bounded and measurable $\mathcal{I}_1$ function. Then (i) $F_n : \mathbb{S}_{N_n} \to \mathbb{R}$ as defined by (13) is an $\mathcal{I}_1$ function and (ii) $F : \mathbb{S}_1 \to \mathbb{R}$ as defined by (12) is an $\mathcal{S}$ function.*

The complete proof of this result is in the Appendix. The proof requires the following lemma, which also conveys much of the intuition of the result.

Lemma 1 *Let $f : \mathbb{S} \to \mathbb{R}$ be an $\mathcal{I}_1$ function and let s_n^1 and s_n^2 be elements of $\mathbb{S}_n$. The function $F_n : \mathbb{S}_{N_n} \to \mathbb{R}$, defined by*

$$F_n(s_{N_n}) = f(s_{N_n}, s_n^1) + f(s_{N_n}, s_n^2), \tag{16}$$

is an $\mathcal{I}_1$ function.

Proof: Let $K \subset N_n$ and suppose $s''_K > s'_K$ (with $s''_1 = s'_1$ if $1 \in K$). Suppose $F_n(s^*_{N_n \setminus K}, s''_K) > 0$ and $F_n(s^*_{N_n \setminus K}, s'_K) < 0$. We need to show that

$$-\frac{F_n(s^*_{N_n \setminus K}, s'_K)}{F_n(s^*_{N_n \setminus K}, s''_K)} \geq -\frac{F_n(s^{**}_{N_n \setminus K}, s'_K)}{F_n(s^{**}_{N_n \setminus K}, s''_K)} \text{ if } s^{**}_{N_n \setminus K} > s^*_{N_n \setminus K}. \tag{17}$$

Set the left hand side of (17) equal to Q and define ϕ as the map from $s_{N_n \setminus K}$ to $QF_n(s_{N_n \setminus K}, s''_K) + F_n(s_{N_n \setminus K}, s'_K)$. Using (16), we may write

$$\phi(s_{N_n \setminus K}) = Q[f(s_{N_n \setminus K}, s''_K, s_n^1) + f(s_{N_n \setminus K}, s''_K, s_n^2)] + [f(s_{N_n \setminus K}, s'_K, s_n^1) + f(s_{N_n \setminus K}, s'_K, s_n^2)]. \tag{18}$$

By the definition of Q, we have $\phi(s^*_{N_n \setminus K}) = 0$. Notice that since F_n is an $\mathcal{S}$ function, $F_n(s^{**}_{N_n \setminus K}, s''_K) > 0$, so that (17) holds if $\phi(s^{**}_{N_n \setminus K}) \geq 0$. The latter is true if we can construct two $\mathcal{S}$ functions A^1 and A^2 (of $s_{N_n \setminus K}$) such that (i) $\phi(s_{N_n \setminus K}) = A^1(s_{N_n \setminus K}) + A^2(s_{N_n \setminus K})$, (ii) $A^1(s^*_{N_n \setminus K}) = A^2(s^*_{N \setminus K}) = 0$, and (iii) A^1 and A^2 are $\mathcal{S}$ functions.

Since $F_n(s^*_{N_n \setminus K}, s''_K) = f(s^*_{N_n \setminus K}, s''_K, s_n^1) + f(s^*_{N_n \setminus K}, s''_K, s_n^2) > 0$ and f is an $\mathcal{S}$ function, we must have $f(s^*_{N_n \setminus K}, s''_K, s_n^2) > 0$, while $f(s^*_{N_n \setminus K}, s''_K, s_n^1)$ may be negative (Case 1) or nonnegative (Case 2).

Case 1. Choose $\alpha \in (0, 1)$ so that A^1 defined by

$$A^1(s_{N_n \setminus K}) = Qf(s_{N_n \setminus K}, s''_K, s_n^1) + \alpha Qf(s_{N_n \setminus K}, s''_K, s_n^2)$$

satisfies $A^1(s^*_{N_n \setminus K}) = 0$. Define the function A^2 by

$$A^2(s_{N_n \setminus K}) = (1 - \alpha)Qf(s_{N_n \setminus K}, s''_K, s_n^2) + [f(s_{N_n \setminus K}, s'_K, s_n^1) + f(s_{N_n \setminus K}, s'_K, s_n^2)].$$

With these definitions, it is clear that (i) and (ii) are true (see (18)). Furthermore, since f is an $\mathcal{I}_1$ function, A^1 and A^2 are both sums of $\mathcal{S}$-summable functions (of $s_{N_n\setminus K}$) and are thus $\mathcal{S}$ functions, i.e. (iii) holds as well.[8]

Case 2. Assume, for now, that

$$Qf(s^*_{N_n\setminus K}, s''_K, s^2_n) + f(s^*_{N_n\setminus K}, s'_K, s^2_n) \geq 0. \tag{19}$$

Since $f(s^*_{N_n\setminus K}, s''_K, s^1_n) \geq 0$, $F_n(s^*_{N_n\setminus K}, s'_K) = f(s^*_{N_n\setminus K}, s'_K, s^2_n) + f(s^*_{N_n\setminus K}, s'_K, s^1_n) < 0$, and $\phi(s^*_{N_n\setminus K}) = 0$, we know that there exists $\beta \in [0,1]$ such that the function A^1 defined by

$$A^1(s_{N_n\setminus K}) = Qf(s_{N_n\setminus K}, s''_K, s^2_n) + f(s_{N_n\setminus K}, s'_K, s^2_n) + \beta f(s_{N_n\setminus K}, s'_K, s^1_n)$$

satisfies $A^1(s^*_{N_n\setminus K}) = 0$. Define the function A^2 by

$$A^2(s_{N_n\setminus K}) = Qf(s_{N_n\setminus K}, s''_K, s^1_n) + (1-\beta) f(s_{N_n\setminus K}, s'_K, s^1_n).$$

Clearly, (i) and (ii) holds by construction (see (18)). Lastly, (iii) holds because f is an $\mathcal{I}_1$ function, so A^1 and A^2 are both sums of $\mathcal{S}$-summable functions (of $s_{N_n\setminus K}$).

It remains for us to show that (19) holds. Suppose it does not; since $f(s^*_{N_n\setminus K}, s''_K, s^2_n) > 0$, we must have $f(s^*_{N_n\setminus K}, s'_K, s^2_n) < 0$. This in turn implies that $f(s^*_{N_n\setminus K}, s'_K, s^1_n) < 0$ (because f is an $\mathcal{S}$ function) and hence $f(s^*_{N_n\setminus K}, s''_K, s^1_n) > 0$ (because $\phi(s^*_{N_n\setminus K}) = 0$). We also have

$$Q < -\frac{f(s^*_{N_n\setminus K}, s'_K, s^2_n)}{f(s^*_{N_n\setminus K}, s''_K, s^2_n)} \leq -\frac{f(s^*_{N_n\setminus K}, s'_K, s^1_n)}{f(s^*_{N_n\setminus K}, s''_K, s^1_n)}. \tag{20}$$

The first inequality follows from the violation of (19). The second from the fact that $f(\cdot; s^*_{N_n\setminus K}, s''_K) \sim f(\cdot; s^*_{N_n\setminus K}, s'_K)$ and the numerator and denominator of the right-most ratio are negative and positive respectively. Using (20) and the formula for ϕ (see (18)), we obtain $\phi(s^*_{N_n\setminus K}) < 0$, which is a contradiction. **QED**

[8] Note that the decomposition makes careful use of the $\mathcal{I}_1$ property of f to ensure that A_1 and A_2 are sums of $\sim$-related functions.

Theorem 3 is related to the well-known result that logsupermodularity is preserved under integration (see Karlin and Rinott (1980)); for applications in economics see Jewitt (1991), Gollier (2001) and Athey (2001, 2002)). This property is an easy consequence of Theorem 3.

Corollary 2 *Let X_i (for $i = 1, 2, ..., m$) and Y_j (for $j = 1, 2, ..., n$) be measurable subsets of $\mathbb{R}$ and suppose that the function $\phi : X \times Y \to \mathbb{R}$ (where $X = \Pi_{i=1}^m X_i$ and $Y = \Pi_{j=1}^n Y_j$) is uniformly bounded and measurable with respect to $y \in Y$. If ϕ is logsupermodular in (x, y), then the function Φ, defined by $\Phi(x) = \int_Y \phi(x, y)dy$ is also a logsupermodular function.*

Proof: Let $K \subset M = \{1, 2, ...m\}$, and suppose $a'' > a'$, for a'' and a' in $\Pi_{i \in K} X_i$. Let $b^{**} > b^*$ be two vectors in $\Pi_{i \in M \setminus K} X_i$. Suppose $\Phi(b^*, a'') = Q\, \Phi(b^*, a')$. This means that

$$\int_Y [\phi(b^*, a'', y) - Q\phi(b^*, a', y)]\, dy = 0. \tag{21}$$

Define the function $G : \Pi_{i \in M \setminus K} X_i \to \mathbb{R}$ by $G(b) = \int_Y [\phi(b, a'', y) - Q\phi(b, a', y)]\, dy$. Note that the integrand may be written as

$$\left[\frac{\phi(b, a'', y)}{\phi(b, a', y)} - Q \right] \phi(b, a', y).$$

The term in the square brackets is increasing in (b, y) (because ϕ is logsupermodular); so the integrand is the product of an increasing function of (b, y) and a logsupermodular function of (b, y). Therefore, it is an $\mathcal{I}$ function and Theorem 3 guarantees that G is an $\mathcal{S}$ function. Since $G(b^*) = 0$ (by (21)), we obtain $G(b^{**}) \geq 0$. The latter is equivalent to

$$\frac{\Phi(b^{**}, a'')}{\Phi(b^{**}, a')} \geq Q = \frac{\Phi(b^*, a'')}{\Phi(b^*, a')},$$

which establishes the logsupermodularity of Φ. **QED**

We now present two technical results, both of which are useful in Section 7, where we establish the monotonicity of bidding strategies in first-price auctions. The first result is a generalization of Theorem 2(ii).

Proposition 7 *Let $f : \mathbb{S} \to \mathbb{R}$ be a bounded and measurable $\mathcal{I}_1$ function and suppose that $g \sim f(\cdot; s'_{N_1})$ for every $s'_{N_1} \in \mathbb{S}_{N_1}$, where $g : \mathbb{S}_1 \to \mathbb{R}$ is an $\mathcal{S}$ function. Then $g \sim F$, where F is defined by (12).*

We pointed out in Section 3 that the single crossing property was preserved after domain coarsening. This feature is also true of the integrable single crossing property. Given an element $\hat{s}_n$ in the interior of $\mathbb{S}_n = [\underline{s}_n, \bar{s}_n]$, we define the function $\bar{F}_n : \mathbb{S}_{N_n} \times \{0,1\} \to \mathbb{R}$ by

$$\bar{F}_n(s_{N_n}, s_n) = \begin{cases} \int_{[\underline{s}_n, \hat{s}_n]} f(s_{N_n}, z_n) dz_n & \text{if } s_n = 0 \\ \int_{(\hat{s}_n, \bar{s}_n]} f(s_{N_n}, z_n) dz_n & \text{if } s_n = 1 \end{cases} \tag{22}$$

Proposition 8 (stated below and proved in the Appendix) says that $\bar{F}_n$ is an $\mathcal{I}_1$ function. Carrying this observation further, let $\hat{s}_i$ be an element in the interior of the closed interval $\mathbb{S}_i$, for $i = 2, 3, ..., n$ and define the function $\bar{F} : \mathbb{S}_1 \times \{0,1\}^{n-1} \to \mathbb{R}$ by

$$\bar{F}(s_1, s_2, .., s_n) = \int_{\mathbb{S}'_2} \int_{\mathbb{S}'_3} ... \int_{\mathbb{S}'_n} f(s_1, z_2, ..., z_{n-1}, z_n)\, dz_2\, dz_3 ... dz_n \text{ where} \tag{23}$$

$$S'_i = \begin{cases} [\underline{s}_i, \hat{s}_i] & \text{if } s_i = 0 \text{ and} \\ (\hat{s}_i, \bar{s}_i] & \text{if } s_i = 1. \end{cases} \tag{24}$$

The function $\bar{F}$ is also an $\mathcal{I}_1$ function if f is an $\mathcal{I}_1$ function.

Proposition 8 *Let $f : \mathbb{S} = [\underline{s}_i, \bar{s}_i]^n \to R$ be a bounded and measurable $\mathcal{I}_1$ function. Then $\bar{F}_n$ (as defined by (22)) and $\bar{F}$ (as defined by (23) and (24)) are also $\mathcal{I}_1$ functions.*

6. Monotone decision rules

In this section, we consider the application of Theorem 3 to an important class of optimization problems, first raised in our discussion of *Problem 2* (in Section 2).

Consider an agent operating under uncertainty who chooses an action x (in $X \subset \mathbb{R}$) to maximize his expected utility, given by

$$V_1(x;s_1) = \int_{\mathbb{S}_{N_1}} v_1(x;s_1,s_2,...,s_n)\lambda(s_{N_1}|s_1)ds_{N_1}. \tag{25}$$

The agent's realized utility, $v_1(x;s_1,s_2,...,s_n)$ depends on some variable s_1 observed by the agent and also on other unobserved state variables $(s_2,s_3,...,s_n)$. Assume that $\mathbb{S}_i$ is a compact interval. If this agent is making a decision in the context of a Bayesian game, s_j (for $j \in N_1$) may be a signal observed by the jth player but not this agent. The distribution of s_{N_1} conditional on observing s_1 is given by the density $\lambda(s_{N_1}|s_1)$.

Suppose that the random variables $\{s_i\}_{i\in N}$ are *affiliated* in the sense that the joint distribution of $(s_1,s_2,...,s_n)$ admits a logsupermodular density function λ. It is straightforward to show that the density function of the conditional distribution, i.e., $\lambda(s_{N_1}|s_1)$, is also logsupermodular in $(s_1,s_2,...,s_n)$. Suppose, in addition, that for any two actions $x'' > x'$,

$$\Delta(s) = v_1(x'';s) - v_1(x';s) \tag{26}$$

is an $\mathcal{I}_1$ function. Then Theorem 3 guarantees $V_1(x'';\cdot) - V_1(x';\cdot)$ is an $\mathcal{S}$ function; in other words, the family $\{V_1(\cdot;s)\}_{s_1\in\mathbb{S}_1}$ obeys single crossing differences. This in turn guarantees that the agent's optimal action increases with the signal he receives (by Theorem 1). The next result states these observations formally.

Theorem 4 *Suppose that $\lambda(\cdot|\cdot)$ is logsupermodular and, for any $x'' > x'$, Δ (as defined by (26)) is an $\mathcal{I}_1$ function. Then $\{V_1(\cdot;s_1)\}_{s_1\in\mathbb{S}_1}$ (with V_1 defined by (25)) obeys single crossing differences and* $\text{argmax}_{x\in X} V_1(x;s_1)$ *increases with s_1.*

As an easy application of Theorem 4, suppose v_1 is a logsupermodular function (of $(x;s)$). For any $x'' > x'$, we have

$$\Delta(s) = \left[\frac{v_1(x'';s)}{v_1(x';s)} - 1\right] v_1(x';s).$$

We see that Δ is the product of an increasing function (the term in the square brackets, which is increasing in s because v_1 is logsupermodular) and the logsupermodular function $v_1(x';\cdot)$. Thus Δ is an $\mathcal{I}_1$ (in fact, an $\mathcal{I}$) function and we conclude that $\{V_1(\cdot;s_1)\}_{s_1\in\mathbb{S}_1}$ obeys single crossing differences. So we recover the well-known result that $\text{argmax}_{x\in X} V_1(x;s_1)$ increases with s_1 when v_1 is logsupermodular.[9] Of course, the power of Theorem 4 lies precisely in the fact that it gives a way of guaranteeing monotonic decision rules even when v_1 is not necessarily logsupermodular, as is the case in the applications at the end of this section.

The conditions in Theorem 4 are tight in the following sense: if we wish to guarantee that $\{V_1(\cdot;s)\}_{s_1\in\mathbb{S}_1}$ is a single crossing family whenever $\lambda(\cdot|\cdot)$ is logsupermodular then (subject to some mild regularity conditions) it is *necessary* that Δ be an $\mathcal{I}_1$ function. The next result makes the role of the $\mathcal{I}_1$ property transparent.

Proposition 9 *(i) Suppose that for every $x'' > x'$, Δ (as defined by (26)) is an $\mathcal{I}_1$ function. Then v_1 has the following property: $(\star)$ for any $\alpha\in[0,1]$, $s_1^{**} > s_1^*$ in $\mathbb{S}_1$, vectors $a^{**}\geq a^*$ in $\mathbb{S}_K$ with $K\subset N_1$, and $b^{**}\geq b^*$ in $\mathbb{S}_{N_1\backslash K}$,*

$$\alpha v_1(x'';s_1^*,b^*,a^*)+(1-\alpha)v_1(x'';s_1^*,b^*,a^{**}) \geq (>)\, \alpha v_1(x';s_1^*,b^*,a^*)+(1-\alpha)v_1(x';s_1^*,b^*,a^{**}) \tag{27}$$

$$\Longrightarrow$$

$$\alpha v_1(x'';s_1^{**},b^{**},a^*)+(1-\alpha)v_1(x'';s_1^{**},b^{**},a^{**}) \geq (>)\, \alpha v_1(x';s_1^{**},b^{**},a^*)+(1-\alpha)v_1(x';s_1^{**},b^{**},a^{**}). \tag{28}$$

(ii) Suppose v_1 is continuous in s_1 and that, for some $x''>x'$, Δ is not *an $\mathcal{I}_1$ function on the restricted domain $(\text{Int}\,\mathbb{S}_1)\times\mathbb{S}_{N_1}$. Then v_1 violates $(\star)$.*

The inequalities (27) and (28) require some explanation. The left (right) hand side of (27) is the expected utility of action x'' (x') if, after observing s_1^*, the agent places a probability of α on (s_1^*,b^*,a^*) and $1-\alpha$ on (s_1^*,b^*,a^{**}). The left and right of (28)

[9]For applications of this result in Bayesian games, see Athey (2001).

compares the expected utility of x'' and x' if, after observing the signal s_1^{**} (which is strictly higher than s_1^*), the agent places a probability of α on (s_1^{**}, b^{**}, a^*) and $(1-\alpha)$ on $(s_1^{**}, b^{**}, a^{**})$. Since $(s_1^{**}, b^{**}, a^*) > (s_1^*, b^*, a^*)$ and $(s_1^{**}, b^{**}, a^{**}) > (s_1^*, b^*, a^{**}))$, a higher signal leads to an upward revision of beliefs. Whenever the agent's posterior beliefs vary with the signal s_1 is this way, property $(\star)$ requires that if x'' is preferred to x' at signal s_1^*, then the preference will remain at the higher signal s_1^{**}. Proposition 9 says that property $(\star)$ holds if and only if Δ is an $\mathcal{I}_1$ function. In particular, if Δ is not an $\mathcal{I}_1$ function, then there is an instance where the preference of x'' over x' is reversed when the agent's signal is raised from s_1^* to s_1^{**}.

Proof of Proposition 9: (i) In the case where $a^* = a^{**}$, property $(\star)$ follows from the our assumption that Δ is an $\mathcal{S}$ function. So assume that $a^{**} > a^*$. The inequality (27) may be re-written as $\alpha\Delta(s_1^*, b^*, a^*) + (1-\alpha)\Delta(s_1^*, b^*, a^{**}) \geq (>)\, 0$. Since $\Delta(\cdot, a^*) \sim \Delta(\cdot, a^{**})$, we obtain $\alpha\Delta(s_1^{**}, b^{**}, a^*) + (1-\alpha)\Delta(s_1^{**}, b^{**}, a^{**}) \geq (>) 0$ (by Proposition 1), which is (28).

(ii) Suppose that for some $x'' > x'$, Δ is not an $\mathcal{S}$ function. Then there is $s^{**} > s^*$ such that either (a) $\Delta(s^*) \geq 0$ but $\Delta(s^{**}) < 0$ or (b) $\Delta(s^*) > 0$ and $\Delta(s^{**}) = 0$. Since Δ is continuous in s_1 and s_1^{**} and s_1^* are in the interior of $\mathbb{S}_1$, we may assume that $s_1^{**} > s_1^*$ (in other words, we may exclude the case where $s_1^{**} = s_1^*$). Setting $K = \emptyset$, $\alpha = 1$, $b^* = s_{N_1}^*$ and $b^{**} = s_{N_1}^{**}$, it is clear that we have a violation of property $(\star)$.

Assume now that Δ is an $\mathcal{S}$ function, but it is not an $\mathcal{I}_1$ function because for some $a^{**} > a^*$, with a^* and a^{**} in $\mathbb{S}_K$ and $K \subset N_1$, the $\mathcal{S}$ functions $\Delta(\cdot, a^{**})$ and $\Delta(\cdot, a^*)$ are not $\sim$-related. By Proposition 1, there is $\hat{\alpha}$ such that $\hat{\alpha}\Delta(\cdot, a^{**}) + (1-\hat{\alpha})\Delta(\cdot, a^*)$ is not an $\mathcal{S}$ function. This means that there is $(s_1^{**}, b^{**}) > (s_1^*, b^*)$, with b^{**} and b^* in $\mathbb{S}_{\hat{N}\setminus K}$ such that, either (a$'$) $\hat{\alpha}\Delta(s_1^*, b^*, a^{**}) + (1-\hat{\alpha})\Delta(s_1^*, b^*, a^*) \geq 0$ but $\hat{\alpha}\Delta(s_1^{**}, b^{**}, a^{**}) + (1-\hat{\alpha})\Delta(s_1^{**}, b^{**}, a^*) < 0$ or (b$'$) $\hat{\alpha}\Delta(s_1^*, b^*, a^{**}) + (1-\hat{\alpha})\Delta(s_1^*, b^*, a^*) > 0$ but $\hat{\alpha}\Delta(s_1^{**}, b^{**}, a^{**}) + (1-\hat{\alpha})\Delta(s_1^{**}, b^{**}, a^*) = 0$. Note that because s_1^* and s_1^{**} are in the interior of $\mathbb{S}_1$ and Δ is continuous in s_1, we can always guarantee that $s_1^{**} > s_1^*$. Either (a$'$) or (b$'$) is equivalent to violation of $(\star)$. **QED**

We end this section with two applications of Theorem 4.

Extension of Application 1.

Suppose the monopolist has to make its output decision when both demand and cost conditions are not fully known; we also allow the monopolist to have an additional and stochastic source of income, which we denote by b (drawn from the set $\mathbb{B}$ in $\mathbb{R}$). The monopolist observes a signal k in $\mathbb{K} \subset \mathbb{R}$; the joint distribution of (b, s, t) given k is given by the density function $\mu(\cdot|k)$. We assume that $\mu((b, s, t)|k)$ is logsupermodular in (b, s, t, k)). Loosely speaking, this says that a high signal makes higher realizations of b, s, and t more likely. After observing k, the monopolist chooses x to maximize

$$V(x; k) = \int_{\mathbb{B}\times T\times\mathbb{S}} h\left(\Pi(x; s, t) + b\right) \mu\left((b, s, t)|k\right)\, db\, ds\, dt. \tag{29}$$

The next result identifies conditions under which the monopolist's optimal output choice increases with the signal received.

Proposition 10 *Suppose that* $\{\Pi(\cdot; s, t)\}_{(s,t)\in\mathbb{S}\times T}$ *is given by (9) and has the following properties: (i)* C *is increasing* x*, decreasing in* s*, and* $\{C(\cdot; s)\}_{s\in\mathbb{S}}$ *obeys decreasing differences, and (ii)* P *is decreasing in* x *and increasing in* t *and* $\{P(\cdot; t)\}_{t\in T}$ *obeys increasing differences. In addition, (iii) suppose that* $h : \mathbb{R} \to \mathbb{R}$ *is twice differentiable, with* $h' > 0$ *and obeys DARA and that (iv)* $\mu(\cdot|\cdot)$ *is a logsupermodular function. Then* $\{V(\cdot; k)\}_{k\in\mathbb{K}}$ *(as defined by (29)) obeys single crossing differences, and* $\text{argmax}_{x\in X} V(x; k)$ *increases with* k*.*

Proof: Notice that the monopolist's realized income at output x, $\Pi(x; s, t) + b$, is increasing in (b, s, t). Notice also that condition (iii) in this proposition coincides with condition (iii) in Proposition 5, while condition (ii) in this proposition is stronger than (ii) in Proposition 5.[10] Given this strengthening, in addition to $\{\Pi(\cdot; s, t)\}_{s\in\mathbb{S}}$ obeying

[10] See remark following Proposition 5.

increasing differences (for any given t), we also have $\{\Pi(\cdot; s, t)\}_{t \in T}$ obeying increasing differences (for any given s). Proposition 5 tells us that $\{\Delta(\cdot, t)\}_{t \in T}$ (defined by (10)) is an $\mathcal{S}$-summable family; given our strengthened assumptions, the same argument we made there will guarantee that $\{\Delta(s, \cdot)\}_{s \in \mathbb{S}}$ is an $\mathcal{S}$-summable family. These properties, along with condition (iv), guarantee that the map from (b, s, t, k) to $\Delta(s, t)\mu((b, s, t)|k)$ is an $\mathcal{I}$ function. So it follows from Theorem 4 that $\{V(\cdot; k)\}_{k \in \mathbb{K}}$ obeys single crossing differences and $\text{argmax}_{x \in X} V(x; k)$ increases with k. **QED**

Application 2: Bertrand oligopoly with differentiated products

Consider a Bertrand oligopoly with n firms, each selling a single differentiated product. We focus our discussion on firm 1 (the situation of the other firms being analogous). Firm 1 has a constant unit cost of production of c_1; the demand for its output if it charges price p_1 and the other firms charge p_{N_1} (for their respective products) is given by $D_1(p_1, p_{N_1}; s_1)$, where s_1 is some parameter affecting demand that is observed by firm 1. In general, firm j observes s_j but not s_k for $k \neq j$. At the price vector $p = (p_1, p_{N_1})$ and the parameter s_1, firm 1's profit is $\Pi_1(p_1, p_{N_1}; s_1) = (p_1 - c_1)D_1(p_1, p_{N_1}; s_1)$. Suppose that firm $j \neq 1$ charges the price $\psi_j(s_j)$ whenever it observes s_j. If so, Firm 1 chooses p_1 to maximize its expected utility

$$V_1(p_1; s_1) = \int_{\mathbb{S}_{N_1}} h_1(\Pi_1(p_1, [\psi_j(s_j)]_{j \in N_1}; s_1)\lambda(s_{N_1}|s_1)\, ds_{N_1},$$

where h_1 is the firm's Bernoulli utility function and $\lambda(\cdot|s_1)$ is the distribution of s_{N_1}, conditional on observing s_1.

We would like to find conditions under which there exists a Bayesian Nash equilibrium to this game. We know from Athey (2001) that (subject to some mild regularity conditions) a Bayesian Nash equilibrium (with equilibrium decision rules that are increasing in the signal) exists if each firm has an *optimal* decision rule that is increasing, given that all other firms are playing increasing decision rules.[11] Therefore we are interested in the primitive conditions under which $\text{argmax}_{p_1>c_1} V_1(p_1; s_1)$ is increasing in s_1, given that ψ_j are increasing functions.

To address this issue, consider in the first instance the case where the agent is risk neutral, so h_1 is the identity function. Suppose that D_1 is a logsupermodular function of $(p_1, p_{N_1}; s_1)$. This condition has a very simple interpretation in terms of the elasticity of demand. Define

$$\epsilon_i(p; s_1) = \frac{p_i}{D_1(p; s_1)} \frac{\partial D_1}{\partial p_i}(p; s_1)\,;$$

the logsupermodularity of D_1 is equivalent to ϵ_i being increasing in s_1 and in p_k for $k \neq i$. It is straightforward to check that if D_1 is (i) increasing in p_{N_1} (so an increase in the price charged by firm $k \neq 1$ raises the demand for firm 1's product), (ii) logsupermodular, and (iii) ψ_j is increasing for all $j \geq 2$, then $\Pi_1(p_1, [\psi_j(s_j)]_{j \in N_1}; s_1)$ is logsupermodular in $(p_1; s)$. If, in addition, $\lambda(\cdot|\cdot)$ is logsupermodular, then V_1 is logsupermodular in $(p_1; s_1)$ and so we conclude that $\text{argmax}_{p_1>c_1} V_1(p_1; s_1)$ increasing in s_1. This result is generalized in the next proposition, which uses Proposition 4 and Theorem 4 to consider the case where the firm is not necessarily risk neutral.

Proposition 11 *Suppose that ψ_j is increasing for all $j \in N_1$, $\lambda(\cdot|\cdot)$ is logsupermodular, and D_1 is increasing p_{N_1} and in s_1, with ϵ_1 increasing in s_1 and in p_k for all $k \neq i$. Then* $\text{argmax}_{p_1>c_1} V_1(p_1; s_1)$ *is increasing in s_1 if any of the following conditions hold:*
(A) $h_1(z) = \ln z$, *i.e., the coefficient of relative risk aversion is identically 1;*
(B) *the coefficient of relative risk aversion is bounded above by 1 and is decreasing and, for $i \neq 1$, ϵ_i is increasing in s_1 and in p_k for all $k \neq \{i, 1\}$;*
(C) *the coefficient of relative risk aversion is bounded below by 1 and is decreasing and, for $i \neq 1$, ϵ_i is decreasing in s_1 and in p_k for all $k \neq \{i, 1\}$.*[12]

[11] For generalizations of Athey's work, see McAdams (2003), Van Zandt and Vives (2007), and Reny (2009).

[12] The conditions on ϵ_1, together with the (B) conditions on ϵ_i, for $i \neq 1$, are equivalent to the

Proof: We may write $h_1(\Pi_1(p_1, (\psi_j(s_j))_{j\in N_1}; s_1))$ as $\tilde{h}_1(\tilde{\pi}(p_1; s))$ where $\tilde{h}_1(\cdot) = h_1(\exp(\cdot))$ and $\tilde{\pi}(p_1; s) = \ln \Pi_1(p_1, (\psi_j(s_j))_{j\in N_1}; s_1)$. For any $p_1'' > p_1'$ we define Δ by $\Delta(s) = \tilde{h}(\tilde{\pi}(p_1''; s)) - \tilde{h}(\tilde{\pi}(p_1'; s))$. By Theorem 4, we need only show that Δ is an $\mathcal{I}$ function. Note that the conditions (in particular the conditions on ϵ_1) guarantee the property (p1): $\tilde{\pi}(p_1''; s) - \tilde{\pi}(p_1'; s)$ is increasing in s. We also have the property (p2): $\tilde{\pi}$ is increasing in $s = (s_{N\setminus K}, s_K)$.

Case (A) is the easiest of the three cases. We have $\Delta(s) = \tilde{\pi}(p_1''; s) - \tilde{\pi}(p_1'; s)$, so Δ is certainly as $\mathcal{I}$ function since it is an increasing function (by (p1)). For cases (B) and (C), we first note that (p1) guarantees that Δ is an $\mathcal{S}$ function. To confirm that Δ is an $\mathcal{I}$ function, we need to check that

$$\Delta(\cdot, s_K'') \sim \Delta(\cdot, s_K'). \tag{30}$$

where $K \subset N$ and $s_K'' > s_K'$. This can be obtain via Proposition 4 (with $T = \{s_K'', s_K'\}$). Consider the assumptions under case (B). Those assumptions guarantee property (p3): for any p_1, $\tilde{\pi}(p_1; s_{N\setminus K}^{**}, s_K) - \tilde{\pi}(p_1; s_{N\setminus K}^{*}, s_K)$ is increasing in s_K, for any $s_{N\setminus K}^{**} > s_{N\setminus K}^{*}$; they also guarantee (p4): $\tilde{h}$ is a convex function with DARA. Properties (p1), (p2), (p3), and (p4) together ensure that conditions (i), (ii), (iii), (iv), and (v-c) in Proposition 4 are satisfied. We conclude that (30) holds.

The assumptions of case (C) guarantee property (p3′): for any p_1, $\tilde{\pi}(p_1; s_{N\setminus K}^{**}, s_K) - \tilde{\pi}(p_1; s_{N\setminus K}^{*}, s_K)$ is decreasing in s_K, for any $s_{N\setminus K}^{**} > s_{N\setminus K}^{*}$; they also guarantee property (p4′): $\tilde{h}$ is concave with DARA. [Note the contrast between (p3) and (p3′) and between (p4) and (p4′).] In this case, conditions (i), (ii), (iii), (iv), and (v-b) in Proposition 4 are satisfied and we obtain (30). **QED**

logsupermodularity of D_1. Note also that if h is linear then the coefficient of relative risk aversion equals 0, so the case we discussed just before stating this proposition is covered under (B).

7. Equilibrium existence in first price auctions

Consider a first-price auction with n bidders forming the set N. Before making his bid, bidder $i \in N$ receives a private signal s_i in $\mathbb{S}_i = [0,1]$. The signals $s = (s_1, s_2, ..., s_n)$ have a joint distribution governed by the density function λ. Bidder i submits a sealed bid from $\mathbb{B}_i = \{\ell\} \cup [r_i, \infty)$, where $\ell < r_i$. The bidder submitting the highest bid wins the object, with ties broken uniformly and randomly. The bid ℓ should be interpreted as a decision not to participate in the auction; its payoff to the bidder is normalized at 0, irrespective of the signal. Bidder i faces a reserve price of r_i and if he makes a *serious bid* b, i.e., a bid $b \geq r_i$, then his payoff upon winning the object is $u_i(b; s)$. We say that u_i is *regular* if it is measurable in $(b; s)$, bounded in s for any given b, continuous in b for any given s, there is $\tilde{b}_i$ such that $u_i(b; s) < 0$ for all s if $b > \tilde{b}_i$, and $u_i(\ell; s) = 0$ for all s. Note that $\tilde{b}_i$ constitutes a bid that is so high that bidder i's payoff is always negative, no matter what state s is realized.

At the interim stage, when the i has observed his signal but not that of the other bidders, he decides on his bid by maximizing expected utility. Bidder i's *bidding strategy* is a map from $\mathbb{S}_i$ to $\mathbb{B}_i$. We prove the following equilibrium existence result.

Theorem 5 *A Bayesian-Nash equilibrium with increasing bidding strategies exists when λ is strictly positive in $[0,1]^n$ and logsupermodular and when each bidder i has a regular payoff function $u_i : \mathbb{B}_i \times [0,1]^n \to \mathbb{R}$ that also obeys the following conditions:*
(a) for any serious bid b, $u_i(b; s)$ is increasing in s and strictly increasing in s_i;
(b) for any serious bids $b'' > b'$, $u_i(b''; s) < u_i(b'; s)$ for all s; and
(c) for any serious bids $b'' > b'$, $u_i(b''; \cdot, s'_K) - u_i(b'; \cdot, s'_K) \sim u_i(b''; \cdot, s''_K)$ where $K \subset \{1, 2, ..., n\} \setminus \{i\}$ and $s''_K > s'_K$.

Remark: Condition (b) implies that $u_i(b''; \cdot, s'_K) - u_i(b'; \cdot, s'_K) < 0$, so (c) could be equivalently stated as the following: *for any serious bids $b'' > b'$, $u_i(b''; \cdot, s'_K) - u_i(b'; \cdot, s'_K) \preceq u_i(b''; \cdot, s''_K)$ where $K \subset \{1, 2, ..., n\} \setminus \{i\}$ and $s''_K > s'_K$.*

Our result is closely related to Reny and Zamir (2004); the main result in their paper (Theorem 2.1) has precisely the same setup and conclusion as Theorem 5, except that instead of conditions (a), (b) and (c), they require (a) and the following condition: (**RZ**) *for any serious bids* $b'' > b'$, $u_i(b''; s) - u_i(b'; s)$ *is increasing in* s. To compare the two sets of conditions, first note that (unlike us) Reny and Zamir do not require the bidders payoff to decrease with his bid, i.e., they do not impose condition (b). However, if we assume that (b) is satisfied – and this condition seems innocuous – then (RZ) and (a) clearly imply (c), since both $u_i(b''; s) - u_i(b'; s)$ and $u_i(b'; s)$ will then be increasing in s. In other words, within the class of payoff functions which are decreasing in the bid, our conditions are more general than those in Reny and Zamir (2004). We give two examples to show that our modification of the conditions in Reny and Zamir is significant.

Example 1. Suppose that $u_i(b; s) = u_i(s_i - b)$; this case was considered by Athey (2001) who showed that an equilibrium in increasing bidding strategies exists if u_i is log-concave. Note that the log-concavity of u_i does *not* imply that $u_i(s-b'')-u_i(s-b')$ is increasing in s; indeed, the latter is equivalent to the concavity of u_i. However, Athey's condition does imply that (c) is satisfied, since (c) requires

$$\frac{u_i(s_i - b') - u_i(s_i - b'')}{u_i(s_i - b'')} = \frac{u_i(s_i - b')}{u_i(s_i - b'')} - 1$$

to be decreasing in s_i, which is true if and only if u_i is log-concave.

Example 2. Loosely speaking, condition RZ says that as the state s becomes higher (and thus more favorable), the payoff difference between one bid and another that is lower becomes less significant. This property is violated in those situations where making a higher bid imposes an opportunity cost to the bidder which is higher when the state is more favorable. Consider the case where $u_i(b; s) = (Y_i - b)\phi_i(s) - Y_i$. We interpret $Y_i > 0$ as the agent's overall budget for a project, of which b is spent on acquiring the licence to run the project. The part of the budget left for the successful bidder's operations is $Y_i - b$, from which he derives revenue of $(Y_i - b)\phi_i(s)$. The net payoff to bidder i is thus given by $u_i(b;s)$. In this case, $u_i(b'',s) - u_i(b',s) = (b'-b'')\phi_i(s)$, which is *decreasing in* s if ϕ_i is increasing in s, thus violating RZ.

On the other hand, condition (c) holds so long as ϕ is logsupermodular and increasing in s. Indeed,

$$\begin{aligned} -\frac{u_1(b''; s_{N\backslash K}, s'_K) - u_1(b'; s_{N\backslash K}, s'_K)}{u_1(b''; s_{N\backslash K}, s''_K)} &= \frac{(b''-b')\phi(s_{N\backslash K}, s'_K)}{(Y_1 - b'')\phi(s_{N\backslash K}, s''_K) - Y_1} \\ &= \frac{b''-b'}{\left[(Y_1-b'')\frac{\phi(s_{N\backslash K}, s''_K)}{\phi(s_{N\backslash K}, s'_K)} - \frac{Y_1}{\phi(s_{N\backslash K}, s'_K)}\right]}, \end{aligned}$$

which is decreasing in $s_{N\backslash K}$ (when $b'' > b'$ and $s''_K > s'_K$) if ϕ is logsupermodular and increasing in s.

Proof of Theorem 5: The overall structure of our proof is identical to that of Reny and Zamir's proof of Theorem 2.1 in their paper and has two components.

[A] For all bidders i, let $\mathbb{B}'_i$ be a subset of $\mathbb{B}_i$, with the property that for $i \neq j$, $\mathbb{B}'_i \cap \mathbb{B}'_j = \{\ell\}$. We refer to $\{\mathbb{B}'_i\}_{1\leq i\leq n}$ as a set of *nonintersecting bid spaces.* We claim that any first price auction has a Bayesian-Nash equilibrium in increasing bidding strategies whenever the conditions of Theorem 5 are satisfied and the players are required to bid from a set of non-intersecting bid spaces. This claim follows immediately from Athey's (2001) equilibrium existence theorem, provided we can show that when all other bidders play an increasing bidding rule, then player k's optimal bid is increasing in the signal he receives.

To be precise (and focussing first on player 1), suppose bidder i (for $i \neq 1$) is playing the increasing strategy $\bar{b}_i : \mathbb{S}_i \to \mathbb{B}_i$ and denote the payoff of player 1 if he bids b after observing s_1 and when player i (for $i \neq 1$) is playing the strategy $\bar{b}_i$, by $V_1(b; s_1)$. It suffices for us to show the following property:

($\star$) when $\mathbb{B}'_1 \subset \mathbb{B}_1$ is such that $\mathbb{B}'_1 \cap \text{Range}(\bar{b}_i) = \{l\}$ for all $i \neq 1$ then whenever $s_1^{**} > s_1^*$ we have

$$\text{argmax}_{b\in\mathbb{B}'_1} V_1(b; s_1^{**}) \geq \text{argmax}_{b\in\mathbb{B}'_1} V_1(b; s_1^{*}). \tag{31}$$

In other words, if we confine bidder 1's bid space so that his bid never ties with that from another bidder, then his optimal bid must increase with the signal he receives.

[B] Assuming for now that ($\star$) holds, a Bayesian-Nash equilibrium in increasing bidding strategies exists whenever the bidders are required to bid from a set of non-intersecting bid spaces. Reny and Zamir show that this property, together with condition (a) guarantee the following: there exists a sequence of nonintersecting bid spaces $\{\mathbb{B}_i^n\}_{i\in N}$, each with a Bayesian-Nash equilibrium in increasing strategies $\{\tilde{b}_i^n\}_{i\in N}$ such that, for every i, the function $\tilde{b}_i^n : \mathbb{S}_i \to \mathbb{B}_i^n$ has a limit $\hat{b}_i : \mathbb{S}_i \to \mathbb{B}_i$ and $\{\hat{b}_i\}_{i\in N}$ form a Bayesian-Nash equilibrium of increasing strategies of the first price auction (with the original bid spaces). **QED**

It remains for us to prove ($\star$). Note firstly that it really *is* necessary to exclude bids that tie with other bidders; (31) is not generally true if we set $\mathbb{B}_1' = \mathbb{B}_1$ (see Reny and Zamir (2004)). It follows from Theorem 1 that ($\star$) holds if we can show that the collection of expected payoff functions $\{V_1(\cdot; s_1)\}_{s_1\in\mathbb{S}_1}$ (when restricted to the domain $\mathbb{B}_1'$) obeys single crossing differences. However, Reny and Zamir show that this too is not generally true. To prove ($\star$), Reny and Zamir establish a weaker form of single crossing differences that is nonetheless sufficient to guarantee (31). Following them, we say that the family $\{V_1(\cdot; s_1)\}_{s_1\in\mathbb{S}_1}$ obeys *individually rational and tieless* (IRT) *single crossing differences* if for any b'' and b' with $b'' > b' \geq r_1$ such that

$$\Pr(\bar{b}_i(s_i) = b') = \Pr(\bar{b}_i(s_i) = b'') = 0 \text{ for all } i \neq 1, \tag{32}$$

the following condition is satisfied: if $V_1(b''; s_1^*) \geq 0$, and $s_1^{**} > s_1^*$, then

$$V_1(b''; s_1^*) - V_1(b'; s_1^*) \geq (>)\, 0 \Longrightarrow V_1(b''; s_1^{**}) - V_1(b'; s_1^{**}) \geq (>)\, 0 \tag{33}$$

This property is *weaker* than single crossing differences because of the added requirement that $V_1(b''; s_1^*) \geq 0$. Nonetheless, it is straightforward to show that it is sufficient to guarantee (31). The reason for this is because every bidder can choose to opt out (formally, by taking action ℓ), so that $V_1(b''; s_1^*) \geq 0$ is an individual rationality condition that is always satisfied at any serious optimal bid.[13] The next result establishes IRT single crossing differences and thus $(\star)$.

Proposition 12 *Suppose* λ *is strictly positive in* $[0,1]^n$ *and logsupermodular and bidder* 1 *has a regular payoff function* $u_1 : \mathbb{B}_1 \times [0,1]^n \to \mathbb{R}$ *that obeys the following:*
(a′) for any serious bid b, $u_1(b; s)$ *is an* $\mathcal{I}_1$ *function of* s*;*
(b) for any serious bids $b'' > b'$, $u_1(b''; s) < u_1(b'; s)$ *for all* s*; and*
(c) for any serious bids $b'' > b'$, $u_1(b''; \cdot, s'_K) - u_1(b'; \cdot, s'_K) \sim u_1(b''; \cdot, s''_K)$ *where* $K \subseteq N_1$ *and* $s''_K > s'_K$*.*
Then the family $\{V_1(\cdot; s_1)\}_{s_1 \in \mathbb{S}_1}$ *obeys IRT single crossing differences.*

Remark 1: Note that condition (a′) in this proposition is weaker than (a) in Theorem 5. While (a) is not needed in the proof of Proposition 12 it is used by Reny and Zamir in the limiting argument (see part [B] of our proof of Theorem 5).
Remark 2: Proposition 2.3 in Reny and Zamir (2004) has the same conclusion as Proposition 12, but instead of assuming (a'), (b), and (c), it assumes conditions (a) (as described in our Theorem 5) and condition RZ on u_1.
Proof of Proposition 12: We shall sketch out the broad outline of the proof here, leaving the details to the Appendix. Let b'' be a bid from player 1 and suppose that it obeys (32). Define $\bar{s}_i = \sup\{s_i \in \mathbb{S}_i : \bar{b}_i(s_i) < b''\}$. Since $\bar{b}_i$ is increasing, for $s_i < \bar{s}_i$ ($s_i > \bar{s}_i$), we have $\bar{b}_i(s_i) \leq b''$ ($\bar{b}_i(s_i) \geq b''$). Since ties occur with probability zero, and λ is nonzero, we obtain something stronger: $s_i < \bar{s}_i$ ($s_i > \bar{s}_i$) implies that $\bar{b}_i(s_i) < b''$ ($\bar{b}_i(s_i) > b''$). Therefore, player 1 wins the object for certain if player i's realized signal is strictly below $\bar{s}_i$ and whenever player i receives a signal strictly above $\bar{s}_i$,

[13] In effect, Reny and Zamir are exploiting the fact that single crossing differences is not a necessary condition for comparative statics when the constraint set has more than two elements. The interval dominance order studied in Quah and Strulovici (2009) is a weakening of single crossing differences that also exploits this fact.

player 1 will lose. So player 1's expected utility with bid b'' if he receives signal s_1 is

$$V_1(b''; s_1) = \int_0^{\bar{s}_2} \int_0^{\bar{s}_3} ... \int_0^{\bar{s}_n} u_1(b''; s_1, s_{N\setminus 1})\lambda(s_{N\setminus 1}|s_1)\, ds_2\, ds_3 ...ds_n \tag{34}$$

Let b' be another possible bid from player 1, with $b'' > b'$. Since $V_1(b''; \cdot)$ is an $\mathcal{S}$ function, if $V_1(b''; s_1^*) \geq (>)\, 0$ we also have $V_1(b''; s_1^{**}) \geq (>)\, 0$. Thus (33) must be true if $V_1(b'; s^{**}) < 0$, if $b' = \ell$ or, more generally, if b' is such that b' is a highest bid with probability zero (whereupon $V_1(b'; s_1^*) = V_1(b'; s_1^{**}) = 0$); (33). So we can concentrate on establishing (33) in the case where $V_1(b''; s_1^*) \geq 0$, $V_1(b'; s^{**}) \geq 0$, and $b' > \ell$, with b' being a highest bid with positive probability. Thus, $\hat{s}_i > 0$, where $\hat{s}_i$ is the signal such that for all $s_i < \hat{s}_i$, $\bar{b}_i(s_i) < b'$ and for all $s_i > \hat{s}_i$, $\bar{b}_i(s_i) > b'$. It is clear that $\hat{s}_i \leq \bar{s}_i$.

We define the function Δ by $\Delta(s_1) = V_1(b'', s_1) - V_1(b', s_1)$; therefore

$$\Delta(s_1) = \int_0^{\bar{s}_2} \int_0^{\bar{s}_3} ... \int_0^{\bar{s}_n} \delta(s_1, s_{N_1})\, ds_2\, ds_3 ...ds_n$$

where $\delta : [0, 1] \times [0, \bar{s}_2] \times .... \times [0, \bar{s}_n] \to \mathbb{R}$ is given by

$$\delta(s) = \begin{cases} [u_1(b''; s_1, s_{N_1}) - u_1(b'; s_1, s_{N_1})]\lambda(s_{N_1}|s_1) & \text{if } s_i \leq \hat{s}_i \text{ for } i = 2, 3, ..., n \\ u_1(b''; s_1, s_{N_1})\lambda(s_{N_1}|s_1) & \text{otherwise} \end{cases} \tag{35}$$

Clearly, the problem of showing (33) can be re-cast as showing

$$\Delta(s_1^*) \geq (>)0 \implies \Delta(s_1^{**}) \geq (>)0. \tag{36}$$

We define the function $\bar{\delta} : \mathbb{S}_1 \times \{0, 1\}^{n-1} \to \mathbb{R}$ by

$$\bar{\delta}(s_1, a_2, .., a_n) = \int_{\mathbb{S}'_2} \int_{\mathbb{S}'_3} ... \int_{\mathbb{S}'_n} \delta(s_1, s_{N_1})\, ds_2\, ds_3 ...ds_n \text{ where} \tag{37}$$

$$S'_i = \begin{cases} [0, \hat{s}_i] & \text{if } a_i = 0 \text{ and} \\ (\hat{s}_i, \bar{s}_i] & \text{if } a_i = 1. \end{cases} \tag{38}$$

We may think of a typical element $(a_2, a_3, ..., a_n)$ in $\{0, 1\}^{n-1}$ as representing a particular subset of $\Pi_{i=1}^{n-1}[0, \bar{s}_i]$. Reny and Zamir (2004) refer to these subsets as *cells*). The collection $\{(a_2, a_3, ..., a_n) : a_i = 0, 1\} = \{0, 1\}^{n-1}$ represents a partition of $\Pi_{i=1}^{n}[0, \bar{s}_i]$ and the function $\bar{\delta}$ gives the integral of δ over each cell. To simplify our notation, from this point on, we shall denote the vector $(a_2, a_3, ..., a_n)$ by a. Observe that

$$\sum_{a \in \{0,1\}^{n-1}} \bar{\delta}(s_1, a) = \Delta(s_1). \tag{39}$$

The function $\bar{u}_1(b''; \cdot) : \mathbb{S}_1 \times \{0, 1\}^{n-1} \to \mathbb{R}$ is defined in a similar way to $\bar{\delta}$, but with $\delta(s_1, s_{N_1})$ (in 37) replaced with $u_1(b''; s_1, s_{N_1})\lambda(s_{N_1}|s_1)$. Note that

$$\bar{\delta}(s_1, a) = \bar{u}_1(b'', s_1, a) \text{ for } a > 0 \text{ and} \tag{40}$$

$$\bar{\delta}(s_1, 0) = \bar{u}_1(b''; s_1, 0) - V_1(b'; s_1) \tag{41}$$

Condition (b) guarantees that $\bar{\delta}(s_1, 0) < 0$. Like $u_1(b''; \cdot)$, $\bar{u}_1(b''; \cdot)$ is an $\mathcal{I}_1$ function (by Proposition 8); we also have

$$\sum_{a \in \{0,1\}^{n-1}} \bar{u}_1(b''; s_1, a) = V(b''; s_1). \tag{42}$$

We divide the proof of (36) into two cases: (I) $V_1(b'; s_1^*) \geq 0$; and (II) $V_1(b'; s_1^*) < 0$. For case (I), we show in the Appendix that $\bar{\delta} : \{s_1^*, s_1^{**}\} \times \{0, 1\}^{n-1} \to \mathbb{R}$ is an $\mathcal{I}_1$ function.[14] Provided this is true, (39) and Theorem 3 together guarantee (36).

For case (II), we define the function $\tilde{\delta} : \{s_1^*, s_1^{**}\} \times \{0, 1\}^{n-1} \to \mathbb{R}$ by

$$\tilde{\delta}(s_1^*, a) = \bar{u}_1(b''; s_1^*, a) \text{ for } a \in \{0, 1\}^{n-1} \text{ and} \tag{43}$$

$$\tilde{\delta}(s_1^{**}, a) = \begin{cases} \bar{\delta}(s_1^{**}, 0) - \epsilon & \text{if } a = 0 \text{ and} \\ \bar{\delta}(s_1^{**}, a) & \text{if } a > 0. \end{cases} \tag{44}$$

We show in the Appendix that for positive and sufficiently small ϵ, $\tilde{\delta}$ is an $\mathcal{I}_1$ function. This in turn leads to (36). Indeed, because $V_1(b''; s_1^*) \geq 0$, it follows from (42) and (43) that $\sum_{a \in \{0,1\}^{n-1}} \tilde{\delta}(s_1^*, a) \geq 0$. Since $\tilde{\delta}$ is an $\mathcal{I}_1$ function, Theorem 3 tells us that $\sum_{a \in \{0,1\}^{n-1}} \tilde{\delta}(s_1^{**}; a) \geq 0$. Therefore, we have

$$\Delta(s_1^{**}) = \sum_{a \in \{0,1\}^{n-1}} \tilde{\delta}(s_1^{**}; a) + \epsilon > 0$$

[14] In other words, the restriction of $\bar{\delta}$ to s_1^* and s_1^{**} is an $\mathcal{I}_1$ function.

(by (39) and (44)).[15] **QED**

Appendix

Proof of Theorem 2: To prove (i), consider $s'' > s'$ and suppose the ranges of $f(s'', \cdot)$ and $f(s', \cdot)$ are contained in some bounded interval I. Partition I into disjoint intervals I_h, $h = 1, 2, ..., K$, with a mesh of $1/m$. Denote by $\bar{T}(h^*, h^{**})$ the subset of T such that for $t \in \bar{T}(h^*, h^{**})$, we have $f(s', t) \in I_{h^*}$ and $f(s'', t) \in I_{h^{**}}$. Note that the collection of sets $\bar{T}(h^*, h^{**})$, with h^* and h^{**} ranging between 1 and K form a partition of T. Define the simple functions[16] $f^m(s', \cdot)$ and $f^m(s'', \cdot)$ in the following way: for $t \in \bar{T}(h^*, h^{**})$, choose $f^m(s', t) = f(s', t^0)$ and $f^m(s'', t) = f(s'', t^0)$ for some $t^0 \in \bar{T}(h^*, h^{**})$. It is easy to check that $f^m(s', \cdot)$ tends to $f(s', \cdot)$ pointwise (and, similarly, $f^m(s'', \cdot)$ tends to $f(s'', \cdot)$ pointwise). Proposition 2 guarantees that, if $\int_T f^m(s', t)\, dt > 0$ then $\int_T f^m(s'', t)\, dt > 0$ (since, in this case, the integrals are finite sums). This guarantees that $\int_T f(s', t)\, dt > 0$ implies that $\int_T f(s'', t)\, dt \geq 0$, which is close, but not quite, to what we want.

Note that (i) can be broken into two claims: (a) that $\int_T f(s', t)\, dt \geq 0$ implies $\int_T f(s'', t)\, dt \geq 0$, and (b) that $\int_T f(s', t)\, dt > 0$ implies that $\int_T f(s'', t)\, dt > 0$. To obtain (a), suppose $\int_T f(s', t)\, dt \geq 0$. Note that claim (a) is trivial if $f(s', t) \geq 0$ a.e. Assuming this is not the case, then there must also be $\tilde{t}$ such that $f(s', \tilde{t}) > 0$. Thus, for any $\alpha > 0$, $\int_T f(s', t)\, dt + \alpha f(s', \tilde{t}) > 0$. The argument in the previous paragraph guarantees that $\int_T f(s'', t)\, dt + \alpha f(s'', \tilde{t}) \geq 0$. Since this is true for any $\alpha > 0$, we conclude that $\int_T f(s'', t)\, dt \geq 0$.

To prove (b), first note that the problem is trivial if $f(s'', t) \geq 0$ for all t. So there must be $\hat{t}$ such that $f(s'', \hat{t}) < 0$. Assuming that $\int_T f(s', t)\, dt > 0$, choose $\beta > 0$ sufficiently small, so that $\int_T f(s', t)\, dt + \beta f(s', \hat{t}) > 0$. Therefore, $\int_T f(s'', t)\, dt + \beta f(s'', \hat{t}) \geq 0$. Since $f(s'', \hat{t}) < 0$, we obtain $\int_T f(s'', t)\, dt > 0$.

[15] Note how the introduction of ϵ guarantees that $\Delta(s_1^{**})$ is *strictly* positive.

[16] By a simple function we mean a measurable function that takes finitely many distinct values.

To prove (ii), suppose $g(s') > 0$ and $F(s') = \int_T f(s',t)\,dt < 0$. Let $f^m(s',\cdot)$ and $f^m(s'',\cdot)$ be sequences of simple functions converging pointwise to $f(s',\cdot)$ and $f(s'',\cdot)$ respectively. For m sufficiently large, $F^m(s') \equiv \int_T f^m(s',t)\,dt < 0$. This integral is a finite sum, so by Proposition 2, $-F^m(s')/g(s') \geq -F^m(s'')/g(s'')$. Letting $n \to \infty$, we obtain $-F(s')/g(s') \geq -F(s'')/g(s'')$. Thus, $g \succeq F$. **QED**

The proof of Proposition 4 requires the following lemma.

Lemma 2 *Suppose that h satisfies condition (iii) in Proposition 4. Then for any a_1, a_2, b_1, and b_2 satisfying $a_1 < a_2$, $b_1 < b_2$, $a_1 \leq b_1$, and $a_2 \leq b_2$,*

$$\frac{h(a_2) - h(a_1)}{h(b_2) - h(b_1)} \geq \frac{h(a_2 + w) - h(a_1 + w)}{h(b_2 + w) - h(b_1 + w)} \quad \text{where } w \geq 0. \tag{45}$$

Proof: We prove (45) by the showing that the function F, given by

$$F(x) = \ln\left(h(a_2 + x) - h(a_1 + x)\right) - \ln\left(h(b_2 + x) - h(b_1 + x)\right)$$

is decreasing for $x \geq 0$. Denoting $h' \circ h^{-1}$ by f, the derivative

$$\begin{aligned} \frac{dF}{dx} &= \frac{h'(a_2 + x) - h'(a_1 + x)}{h(a_2 + x) - h(a_1 + x)} - \frac{h'(b_2 + x) - h'(b_1 + x)}{h(b_2 + x) - h(b_1 + x)} \\ &= \frac{f(h(a_2 + x)) - f(h(a_1 + x))}{h(a_2 + x) - h(a_1 + x)} - \frac{f(h(b_2 + x)) - f(h(b_1 + x))}{h(b_2 + x) - h(b_1 + x)} \leq 0, \end{aligned}$$

where the final inequality holds because DARA guarantees f is a convex function and (since h is strictly increasing) $h(a_1 + x) < h(a_2 + x)$, $h(b_1 + x) < h(b_2 + x)$, $h(a_1 + x) \leq h(b_1 + x)$, and $h(a_2 + x) \leq h(b_2 + x)$. **QED**

Proof of Proposition 4: We need to show that $\Delta(\cdot, t'') \sim \Delta(\cdot, t')$. Suppose that $\Delta(s^*, t'') > 0$ and $\Delta(s^*, t') < 0$. This means that $\phi(x''; s^*, t'') - \phi(x'; s^*, t'') > 0$ and $\phi(x''; s^*, t') - \phi(x'; s^*, t') < 0$. Given that $\{\phi(\cdot; s^*, t)\}_{t \in T}$ obeys single crossing differences (condition (ii)), this can only occur if $t' < t''$. Now (iv) guarantees that $\phi(x', s^*, t') \leq \phi(x'; s^*, t'')$, so we obtain

$$\phi(x''; s^*, t') < \phi(x', s^*, t') \leq \phi(x'; s^*, t'') < \phi(x''; s^*, t'').$$

If $s^{**} > s^*$ we have

$$
\begin{aligned}
& -\frac{h(\phi(x''; s^*, t')) - h(\phi(x'; s^*, t'))}{h(\phi(x''; s^*, t'')) - h(\phi(x'; s^*, t''))} \\
\geq\ & \frac{h\left(\phi(x'; s^*, t') + [\phi(x'; s^{**}, t') - \phi(x'; s^*, t')]\right) - h\left(\phi(x''; s^*, t') + [\phi(x'; s^{**}, t') - \phi(x'; s^*, t')]\right)}{h\left(\phi(x''; s^*, t'') + [\phi(x'; s^{**}, t') - \phi(x'; s^*, t')]\right) - h\left(\phi(x'; s^*, t'') + [\phi(x'; s^{**}, t') - \phi(x'; s^*, t')]\right)} \\
\geq\ & \frac{h\left(\phi(x'; s^*, t') + [\phi(x'; s^{**}, t') - \phi(x'; s^*, t')]\right) - h\left(\phi(x''; s^*, t') + [\phi(x'; s^{**}, t') - \phi(x'; s^*, t')]\right)}{h\left(\phi(x''; s^*, t'') + [\phi(x'; s^{**}, t'') - \phi(x'; s^*, t'')]\right) - h\left(\phi(x'; s^*, t'') + [\phi(x'; s^{**}, t'') - \phi(x'; s^*, t'')]\right)} \\
\geq\ & \frac{h\left(\phi(x'; s^*, t') + [\phi(x'; s^{**}, t') - \phi(x'; s^*, t')]\right) - h\left(\phi(x''; s^*, t') + [\phi(x''; s^{**}, t') - \phi(x''; s^*, t')]\right)}{h\left(\phi(x''; s^*, t'') + [\phi(x''; s^{**}, t'') - \phi(x''; s^*, t'')]\right) - h\left(\phi(x'; s^*, t'') + [\phi(x'; s^{**}, t'') - \phi(x'; s^*, t'')]\right)} \\
=\ & -\frac{h(\phi(x''; s^{**}, t')) - h(\phi(x'; s^{**}, t'))}{h(\phi(x''; s^{**}, t'')) - h(\phi(x'; s^{**}, t''))}
\end{aligned}
$$

The first inequality follows from Lemma 2, the second inequality from condition (v) (either versions (a), (b), or (c)), and the third inequality from the fact that h is increasing and that $\{\phi(\cdot; s, t)\}_{s\in\mathbb{S}}$ obeys increasing differences (condition (i)). **QED**

Proof of Proposition 6: Only the "if" part of this claim is nontrivial. A proof with full generality would be heavy with notation, and in a way that may obscure what is going on. Instead of doing that, we shall confine ourselves to a special case that has all the essential features of a complete proof. The reader will have no problems filling in the details for the general case.

Suppose $\mathbb{S} = \Pi_{i=1}^3 \mathbb{S}_i$ and $s_3'' > s_3'$; we claim that $f(\cdot, s_3'') \sim f(\cdot, s_3')$ if the following holds: (i) for any s_1 and $s_3'' > s_3'$, we have $f(\cdot, (s_1, s_3'')) \sim f(\cdot, (s_1, s_3'))$ and (ii) whenever $(s_2'', s_3'') > (s_2', s_3')$, we have $f(\cdot, (s_2'', s_3'')) \sim f(\cdot, (s_2', s_3'))$. (Notice that K has exactly two elements in both (i) and (ii).) It suffices to show that

$$-\frac{f(s_1^*, s_2^*, s_3')}{f(s_1^*, s_2^*, s_3'')} \geq -\frac{f(s_1^{**}, s_2^{**}, s_3')}{f(s_1^{**}, s_2^{**}, s_3'')} \tag{46}$$

when $(s_1^{**}, s_2^{**}) > (s_1^*, s_2^*)$ (see (15)). The left of this inequality is positive by assumption. Since f is an $\mathcal{S}$ function, $f(s_1^{**}, s_2^{**}, s_3'') > 0$ so (46) holds if $f(s_1^{**}, s_2^{**}, s_3') \geq 0$. Therefore (and this is the crucial nontrivial observation in the proof), we may confine ourselves to the case where $f(s_1^{**}, s_2^{**}, s_3') < 0$. We claim that

$$-\frac{f(s_1^*, s_2^*, s_3')}{f(s_1^*, s_2^*, s_3'')} \geq -\frac{f(s_1^{**}, s_2^*, s_3')}{f(s_1^{**}, s_2^*, s_3'')} \geq -\frac{f(s_1^{**}, s_2^{**}, s_3')}{f(s_1^{**}, s_2^{**}, s_3'')}.$$

The first inequality holds because of (ii). Note also that $f(s_1^{**}, s_2^*, s_3') < 0$ (since $f(s_1^{**}, s_2^{**}, s_3') < 0$ and f is an $\mathcal{S}$ function) and $f(s_1^{**}, s_2^*, s_3'') > 0$ (since $f(s_1^*, s_2^*, s_3'') > 0$ and f is an $\mathcal{S}$ function). Therefore, (i) guarantees that the second inequality holds. This establishes (46). **QED**

Our proof of Theorem 3 requires the following extension of Lemma 1.

Lemma 3 *Let $f : \mathbb{S} \to \mathbb{R}$ be an $\mathcal{I}_1$ function and s_n^j (for $j = 1, 2, ..L$) be elements of $\mathbb{S}_n$, with $s_n^j < s_n^{j+1}$. Then $F_n : \mathbb{S}_{N_n} \to \mathbb{R}$ defined by $F_n(s_{N_n}) = \sum_{j=1}^{L} f(s_{N_n}, s_n^j)$ is an $\mathcal{I}_1$ function.*

Proof: By Lemma 1, this lemma is true for $L = 2$. Assuming that it is true for a sum of $L-1$ functions, we show that it is true for a sum of L functions. Consider the function $h : \Pi_{j=1}^{n-1}\mathbb{S}_j \times \{0, 1\} \to \mathbb{R}$ given by $h(s_{N_n}, 0) = \sum_{j=1}^{L-1} f(s_{N_n}, s_n^j)$ and $h(s_{N_n}, 1) = f(s_{N_n}, s_n^L)$. We claim that h is an $\mathcal{I}_1$ function; assuming this, Lemma 1 tells us that the map from s_{N_n} to $h(s_{N_n}, 0) + h(s_{N_n}, 1)$ is an $\mathcal{I}_1$ function, but this map is precisely F_n.

Firstly, note that h is an $\mathcal{S}$ function, since f is an $\mathcal{S}$ function and $h(\cdot, 0)$ is an $\mathcal{I}_1$ – hence $\mathcal{S}$ – function by assumption. To check that (14) holds, let $K \subseteq N_n$ and suppose that $s_K'' > s_K'$, with $s_1'' = s_1'$ if 1 is in K. Since f is an $\mathcal{I}_1$ function, $h(\cdot, s_K'', 1) \sim h(\cdot, s_K', 1)$, and since $h(\cdot, 0)$ is an $\mathcal{I}_1$ function by assumption, we also have $h(\cdot, s_K'', 0) \sim h(\cdot, s_K', 0)$. Given that f is an $\mathcal{I}_1$ function, Proposition 2 tells us that $h(\cdot, s_K'', 1) \sim h(\cdot, s_K', 0)$. By Proposition 6, the only case of condition (14) we still need to show is that $h(s_K'', \cdot) \sim h(s_K', \cdot)$, when $K = N_n$, but in this case (14) is guaranteed by Proposition 3(i). **QED**

Proof of Theorem 3: First note that, by Theorem 2, F_n is an $\mathcal{S}$ function. It remains for us to show that $F_n(\cdot, s_K'') \sim F_n(\cdot, s_K')$ for $K \subseteq \{2, 3, ..., n-1\}$ and $s_K'' > s_K'$. Suppose $s'' > s'$ with $F_n(s', s_K'') > 0$ and $F_n(s', s_K') < 0$. We wish to show that

$$-\frac{F_n(s', s_K')}{F_n(s', s_K'')} \geq -\frac{F_n(s'', s_K')}{F_n(s'', s_K'')}. \tag{47}$$

Using the same procedure as that in Theorem 2, we can construct four sequences of simple functions $f^m(x, y, \cdot)$ defined on $\mathbb{S}_n$, for $x = s'$, s'' and $y = s'_K$, s''_K, such that $f^m(x, y, \cdot) \to f(x, y, \cdot)$ pointwise and $f^m(x, y, s_n) = \sum_{j \in J(m)} \mathbb{I}_{A^m(j)} f(x, y, \tilde{s}_n^j)$, where $\{A^m(j)\}_{j \in J(m)}$ is a collection of disjoint measurable subsets of $\mathbb{S}_n$, $\tilde{s}_n^j \in A^m(j)$ for every $j \in J(m)$, and $\mathbb{I}_{A^m(j)}$ is the indicator function, with $\mathbb{I}_{A^m(j)}(s_n) = 1$ if $s_n \in A^m(j)$ and 0 otherwise. Define $F_n^m(x, y) = \int_{\mathbb{S}_n} f^m(x, y, s_n)\, ds_n$. Note that this integral is a finite sum, so by Lemma 3, $F_n^m(\cdot, s''_K) \sim F_n^m(\cdot, s'_K)$ on the domain $\{s', s''\}$. For m sufficiently large, $F_n^m(s', s''_K) > 0$ and $F_n^m(s', s'_K) < 0$, so we obtain

$$-\frac{F_n^m(s', s'_K)}{F_n^m(s', s''_K)} \geq -\frac{F_n^m(s'', s'_K)}{F_n^m(s'', s''_K)}.$$

Letting m tend to infinity gives (47). **QED**

Proof of Proposition 7: Theorem 2(ii) tell us that, for a given $s'_{N\setminus\{1,n\}}$,

$$F_n(\cdot, s'_{N\setminus\{1,n\}}) \sim g, \tag{48}$$

since $F_n(\cdot, s'_{N\setminus\{1,n\}})$ is the integral over $\{f(\cdot, s'_{N\setminus\{1,n\}}, s_n)\}_{s_n \in \mathbb{S}_n}$, which is an $\mathcal{S}$-summable family of functions of s_1, with $f(\cdot, s'_{N\setminus\{1,n\}}, s_n) \sim g$ for all $s_n \in \mathbb{S}_n$.

Define $F_{n-1} : \Pi_{i=1}^{n-2} \mathbb{S}_i \to \mathbb{R}$ by $F_{n-1}(s_{N\setminus\{n-1,n\}}) = \int_{\mathbb{S}_{n-1}} F_n(s_{N\setminus\{n\}})\, ds_{n-1}$. For a given $s'_{N\setminus\{1,n-1,n\}}$, we claim that $F_{n-1}(\cdot, s'_{N\setminus\{1,n-1,n\}}) \sim g$. Once again, this follows from Theorem 2(ii): note that $F_{n-1}(\cdot, s'_{N\setminus\{1,n-1,n\}})$ is the integral over

$$\{F_n(\cdot, s'_{N\setminus\{1,n-1,n\}}, s_{n-1})\}_{s_{n-1} \in \mathbb{S}_{n-1}},$$

which is an $\mathcal{S}$-summable family of functions of s_1 (by Theorem 3); furthermore, by (48), we have $F_n(\cdot, s'_{N\setminus\{1,n-1,n\}}, s_{n-1}) \sim g$ for every s_{n-1}. Clearly this argument can be repeated until we obtain $F \sim g$. **QED**

Proof of Proposition 8: By Theorem 2, $\bar{F}_n(\cdot, s_n)$ is an $\mathcal{S}$ function. Furthermore, suppose $\bar{F}_n(s_{N_n}, 0) \geq (>)\, 0$. This implies that there is $\hat{z}_n \leq \hat{s}_n$ such that $f(s_{N_n}, \hat{z}_n) \geq (>)\, 0$ which in turn guarantees that $f(s_{N_n}, z_n) \geq (>) 0$ for all $z_n > \hat{s}_n$ (since f is an $\mathcal{S}$ function). It follows that $\bar{F}_n(s_{N_n}, 1) \geq (>) 0$. Therefore $\bar{F}_n$ is an $\mathcal{S}$ function. To check that $\bar{F}_n$ is an $\mathcal{I}_1$ function, we need to check that

$$\bar{F}_n(\cdot, s''_K) \sim \bar{F}_n(\cdot, s'_K), \tag{49}$$

where s''_K and s'_K satisfy the conditions identified in Proposition 6. There are three cases to consider. (i) Suppose $n \in K$ and $s''_n = s'_n$ (in other words, either both equal zero or both equal 1). Then (49) is true since Theorem 3 guarantees that $\bar{F}_n(\cdot, s_n)$ is an $\mathcal{I}_1$ function. (ii) Suppose $n \in K$ and $s''_n = 1 > s'_n = 0$. Note that $f(\cdot; (s'_{K\setminus\{n\}}, z'_n)) \sim f(\cdot, (s''_{K\setminus\{n\}}, z''_n))$ for $z'_n \leq \hat{s}_n < z''_n$. Integrating $f(\cdot, (s'_{K\setminus\{n\}}, z'_n)$ with respect to $z'_n \leq \hat{s}_n$ gives $\bar{F}_n(\cdot, (s'_{K\setminus\{n\}}, 0)) = \bar{F}_n(\cdot, s'_K)$ while integrating $f(\cdot, (s''_{K\setminus\{n\}}, z''_n))$ with respect to $z''_n \geq \hat{s}_n$ gives $\bar{F}_n(\cdot, (s''_{K\setminus\{n\}}, 1)) = \bar{F}_n(\cdot, s''_K)$. In this case, Proposition 2 guarantees (49). (iii) Suppose $n \notin K$; since K has $n-1$ elements, $f(\cdot, s''_K)$ and $f(\cdot, s'_K)$ are both functions of the scalar s_n. By the fact that f is an $\mathcal{I}_1$ function, $f(\cdot, s''_K) \sim f(\cdot, s'_K)$, which guarantees (49) (using Proposition 3(ii)).

By repeating this argument we conclude that $\bar{F}$ is an $\mathcal{I}_1$ function. **QED**

Proof of Theorem 5 continued: For case (I) we still need to show that $\bar{\delta}$ (when restricted to s_1^* and s_1^{**}) is an $\mathcal{I}_1$ function. Given condition (b), $\bar{\delta}(s_1^*, 0) < 0$ and $\bar{\delta}(s_1^{**}, 0) < 0$. Furthermore, $\bar{u}_1(b''; \cdot)$ is an $\mathcal{I}_1$ - hence $\mathcal{S}$ - function. Together these observations tell us that $\bar{\delta}$ (when restricted to s_1^* and s_1^{**}) is an $\mathcal{S}$ function. By Proposition 6, to show that it is an $\mathcal{I}_1$ function, it remains for us to check the following: (A) $\bar{\delta}(s_1, \cdot, a''_K) \sim \bar{\delta}(s_1, \cdot, a'_K)$ for $a''_K > a'_K$, where $K \subset N_1$, and for $s_1 = s_1^*,\ s_1^{**}$ and (B) $\bar{\delta}(\cdot, a'') \sim \bar{\delta}(\cdot, a')$ for $a'' > a'$.

Property (A) says that

$$-\frac{\bar{\delta}(s_1, a'_{N_1\setminus K}, a'_K)}{\bar{\delta}(s_1, a'_{N_1\setminus K}, a''_K)} \geq -\frac{\bar{\delta}(s_1, a''_{N_1\setminus K}, a'_K)}{\bar{\delta}_1(s_1, a''_{N_1\setminus K}, a''_K)} \tag{50}$$

whenever $a''_{N_1\setminus K} > a'_{N_1\setminus K}$ and $\bar{\delta}(s_1, a'_{N_1\setminus K}, a'_K)$ and $\bar{\delta}(s_1, a'_{N_1\setminus K}, a''_K)$ are negative and positive respectively. Since $\bar{u}_1(b'', \cdot)$ is an $\mathcal{I}_1$ function, (50) is true if $(a'_{N_1\setminus K}, a'_K) > 0$ (because of (40)). This leaves us with the case when $(a'_{N_1\setminus K}, a'_K) = 0$. First note that we may assume that $\bar{\delta}(s_1, a''_{N_1\setminus K}, a'_K) = \bar{u}_1(b'', s_1, a''_{N_1\setminus K}, a'_K) < 0$; if not, (50) is always true because its right hand side is negative. This in turn guarantees that $\bar{u}_1(b''; s_1, a'_{N_1\backslash K}, a'_K) = \bar{u}_1(b''; s_1, 0) < 0$ since $\bar{u}_1(b''; \cdot)$ is an $\mathcal{S}$ function. We claim that

$$\begin{aligned}
-\frac{\bar{\delta}(s_1, a'_{N_1\backslash K}, a'_K)}{\bar{\delta}(s_1, a'_{N_1\backslash K}, a''_K)} &= -\frac{\bar{u}_1(b''; s_1, a'_{N_1\backslash K}, a'_K)}{\bar{u}_1(b''; s_1, a'_{N_1\backslash K}, a''_K)} + \frac{V_1(b'; s)}{\bar{u}_1(b''; s_1, a'_{N_1\backslash K}, a''_K)} \\
&\geq -\frac{\bar{u}_1(b''; s_1, a'_{N_1\backslash K}, a'_K)}{\bar{u}_1(b''; s_1, a'_{N_1\backslash K}, a''_K)} \geq -\frac{\bar{u}_1(b''; s_1, a''_{N_1\backslash K}, a'_K)}{\bar{u}_1(b''; s_1, a''_{N_1\backslash K}, a''_K)} \\
&= -\frac{\bar{\delta}(s_1, a''_{N_1\backslash K}, a'_K)}{\bar{\delta}(s_1, a''_{N_1\backslash K}, a''_K)}.
\end{aligned}$$

The first equation follows from (41); the first inequality from the fact that $V_1(b'; s_1) \geq 0$ for $s_1 = s_1^*$, s_1^{**}; the second from the fact that $\bar{u}_1$ is an $\mathcal{I}_1$ function and that $\bar{u}_1(b''; s_1, a'_{N_1\backslash K}, a'_K)$ and $\bar{u}_1(b''; s_1, a'_{N_1\backslash K}, a''_K)$ are negative and positive respectively; and the last equation from (40).

We now turn to the proof of property (B). When $a' > 0$, (B) follows from (40) and the the fact that $\bar{u}_1(b''; \cdot)$ is an $\mathcal{I}_1$ function. So we turn to the case where $a' = 0$. Now suppose $a'' = (1, 1, ..., 1)$; it follows from assumptions (a), (c), and Corollary 7 that, for every $s'_{N_1} < (\hat{s}_2, \hat{s}_3, ..., \hat{s}_n)$, we have $\bar{\delta}(\cdot; s'_{N_1}) \sim \bar{\delta}(\cdot, a'')$. This relation is preserved by integration over all $s'_N < (\hat{s}_2, \hat{s}_3, ..., \hat{s}_n)$ (by assumption (b) and Proposition 7) and so $\bar{\delta}(\cdot, 0) \sim \bar{\delta}(\cdot, a'')$.

Next we consider the case where $a''_i = 0$ for $i = 2, 3, ..., m$ and $a''_i = 1$ for $i > m$. Denote the set $\{1, 2, .., m\}$ by M and a typical vector in $\Pi_{i=1}^m \mathbb{S}_i$ by s_M. Define the function $\alpha : \mathbb{S}_1 \times \Pi_{i=2}^m [0, \hat{s}_i] \times \{0, 1\} \rightarrow \mathbb{R}$ by

$$\begin{aligned}
\alpha(s_M, 0) &= \int_0^{\hat{s}_{m+1}} \int_0^{\hat{s}_{m+2}} ... \int_0^{\hat{s}_n} \delta(s_M, s_{N\backslash M})\, ds_{m+1}\, ds_{m+2} ... ds_n \text{ and} \\
\alpha(s_M, 1) &= \int_{\hat{s}_{m+1}}^{\bar{s}_{m+1}} \int_{\hat{s}_{m+2}}^{\bar{s}_{m+2}} ... \int_{\hat{s}_n}^{\bar{s}_n} \delta(s_M, s_{N\backslash M})\, ds_{m+1}\, ds_{m+2} ... ds_n.
\end{aligned}$$

We claim that α is an $\mathcal{I}_1$ function. If so, $\bar{\alpha} : \mathbb{S}_1 \times \{0, 1\} \rightarrow \mathbb{R}$ defined by

$$\bar{\alpha}(s_1, x) = \int_0^{\hat{s}_2} \int_0^{\hat{s}_3} ... \int_0^{\hat{s}_m} \alpha(s_M, x) ds_2 ds_3 ... ds_m$$

is an $\mathcal{I}_1$ function. Therefore, $\bar{\alpha}(\cdot, 0) \sim \bar{\alpha}(\cdot, 1)$. Since $\bar{\alpha}(s_1, 0) = \bar{\delta}(s_1, 0)$ and $\bar{\alpha}(s_1, 1) = \bar{\delta}(s_1, a'')$, we obtain (B). To see that α is an $\mathcal{I}_1$ function, we first note that $\alpha(\cdot, 0)$

and $\alpha(\cdot, 1)$ are $\mathcal{I}_1$ functions. Furthermore, $\alpha(s_M, 0) < 0$ for any s_M, so it is trivially true that $\alpha(s'_M, \cdot) \sim \alpha(s''_M, \cdot)$ for $s''_M \geq s'_M$. The only property that still needs to be checked is that $\alpha(\cdot, s'_K, 0) \sim \alpha(\cdot, s''_K, 1)$, where $K \subseteq M \setminus \{1\}$, and $s'_K \leq s''_K$. This follows from Proposition 7 and condition (c) since $\delta(\cdot, s'_K, s'_{N\setminus M}) \sim \delta(\cdot, s''_K, s''_{N\setminus M})$ for all $s'_K < s''_K$ and $s'_{N\setminus M} < (\hat{s}_{m+1}, \hat{s}_{m+2}, ..., \hat{s}_n) < s''_{N\setminus M}$.

For case (II) we need to show that $\tilde{\delta}$ is an $\mathcal{I}_1$ function. Note, firstly, that it is an $\mathcal{S}$ function. This is true because $\bar{u}_1(b''; \cdot)$ is an $\mathcal{S}$ function and $\tilde{\delta}(s_1^*, 0)$ and $\tilde{\delta}(s_1^{**}, 0)$ are both strictly negative. Since $\tilde{\delta}(s_1^*, 0) = \bar{u}_1(b''; s_1^*, 0)$ it follows from (41) that

$$\tilde{\delta}(s_1^*; 0) = \bar{\delta}(s_1^*; 0) + V(b'; s_1^*) < 0, \tag{51}$$

while $\tilde{\delta}(s_1^{**}; 0) < \bar{\delta}(s_1^{**}; 0) < 0$. By Lemma 6, we also need to check the following: (A$'$) $\tilde{\delta}_1(s_1, \cdot, a''_K) \sim \tilde{\delta}_1(s_1, \cdot, a'_K)$ for $a''_K > a'_K$, where $K \subset N_1$, and for $s_1 = s_1^*$, s_1^{**} and (B$'$) $\tilde{\delta}(\cdot, a'') \sim \tilde{\delta}_1(\cdot, a')$ for $a'' > a'$. (A$'$) holds for $s_1 = s_1^*$ because $\bar{u}_1(b''; \cdot)$ is an $\mathcal{I}_1$ function; to see that it holds for $s_1 = s_1^{**}$ we need only repeat the argument we gave to establish (A) in Case I above, using the condition that $V_1(b'; s^{**}) \geq 0$.[17] (The inclusion of $-\epsilon$ in this case does not materially alter the argument.) Turning to the proof of (B$'$), if $a'' > a' > 0$, then (B$'$) follows immediately from that fact that $\bar{u}_1(b''; \cdot)$ is an $\mathcal{I}_1$ function. So consider the case where $a' = 0$ and suppose $\tilde{\delta}(s_1^*; a'') > 0$. Then

$$-\frac{\tilde{\delta}(s_1^*, 0)}{\tilde{\delta}(s_1^*, a'')} > -\frac{\bar{\delta}(s_1^*, 0)}{\bar{\delta}(s_1^*, a'')} \geq -\frac{\bar{\delta}(s_1^{**}, 0)}{\bar{\delta}(s_1^{**}, a'')}.$$

The first inequality holds (and it is a *strict* inequality) since $V(b'; s_1^*) < 0$, which guarantees that $-\tilde{\delta}(s_1^*, 0) > \bar{\delta}(s_1^*, 0)$ (see (51)). For the second inequality see the proof of property (B) (in case I). For ϵ sufficiently small, the inequality is preserved (for all possible a'', of which there are only finitely many) when we replace $\bar{\delta}$ with $\tilde{\delta}$, i.e.,

$$-\frac{\tilde{\delta}(s_1^*, 0)}{\tilde{\delta}(s_1^*, a'')} > -\frac{\tilde{\delta}(s_1^{**}, 0)}{\tilde{\delta}(s_1^{**}, a'')},$$

[17] The case where $V_1(b'; s^{**}) < 0$ has already been dealt with in the main part of the paper.

as required by (B′). **QED**

REFERENCES

AMIR, R. (1996): "Cournot Oligopoly and the Theory of Supermodular Games," *Games and Economic Behavior,* 15, 132-148.

ATHEY, S. (2001): "Single Crossing Properties and the Existence of Pure Strategy Equilibria in Games of Incomplete Information," *Econometrica*, 69(4), 861-890.

ATHEY, S. (2002): "Monotone Comparative Statics under Uncertainty," *Quarterly Journal of Economics*, 117(1), 187-223.

GOLLIER, C. (2001): *The Economics of Risk and Time.* Cambridge: MIT Press.

JEWITT, I. (1987): "Risk aversion and the choice between risky prospects: the preservation of comparative statics results," *Review of Economic Studies*, 54, 73-85.

JEWITT, I. (1991): "Applications of likelihood ratio orderings in economics," *IMS Monograph and Lecture Notes*, 19, 174-189.

KARLIN, S. AND Y. RINOTT (1980): "Classes of orderings of measures and related correlation inequalities. I. Multivariate totally positive distributions," *Journal of Multivariate Analysis*, 10, 467-498.

MCADAMS, D. (2003): "Isotone Equilibrium in Games of Incomplete Information", *Econometrica*, 71(4), 1191-1214.

MILGROM, P. (1994): "Comparing Optima: Do simplifying assumptions affect conclusion?" *Journal of Political Economy*, 102(3), 607-615.

MILGROM, P. (2004): *Putting Auction Theory to Work*, Cambridge University Press.

MILGROM, P. AND J. ROBERTS (1990): "Rationalizability, Learning, and Equilibrium in Games with Strategic Complementarities," *Econometrica,* 58(6), 1255-1277.

MILGROM, P. AND C. SHANNON (1994): "Monotone Comparative Statics," *Econometrica*, 62(1), 157-180.

QUAH, J. K.-H. AND B. STRULOVICI (2009): "Comparative Statics, Informativeness, and the Interval Dominance Order," *Econometrica*, 77(6), 1949-1992.

RENY, P. J. (2009): "On the existence of monotone pure strategy equilibria in Bayesian games" (mimeo).

RENY, P. J. AND S. ZAMIR (2004): "On the existence of pure strategy monotone equilibria in asymmetric first-price auctions," *Econometrica*, 72(4), 1105-1125.

SANDMO, A. (1971): "On the theory of the competitive firm under uncertainty," *American Economic Review*, 61, 65-73.

TOPKIS, D. M. (1998): *Supermodularity and Complementarity.* Princeton: Princeton University Press.

VAN ZANDT, T. AND X. VIVES (2007): "Monotone equilibria in Bayesian games of strategic complementarities," *Journal of Economic Theory*, 134(1), 339-360.

VIVES, X. (1990): "Nash Equilibrium with Strategic Complementarities," *Journal of Mathematical Economics*, 19, 305-21.