Supermodular correspondences and Multiprior beliefs

Pawel Dziewulski and John K.-H. Quah

First order stochastic dominance: the EU case

Let $\mathcal{F}$ be the collection of distributions on $S \subset \mathbb{R}$

Let $T \subset \mathbb{R}$ and $\lambda : T \to \mathcal{F}$.

Definition: λ is increasing in first order stochastic dominance if $\lambda(t') \leqslant \lambda(t)$ whenever $t' > t$.

Basic Result 1: λ is FSD-increasing if and only if

$$\int_S \phi(s)\, d\lambda(s,t') \geqslant \int_S \phi(s) d\lambda(s,t)$$

for all increasing functions $\phi : S \to \mathbb{R}$ and $t' > t$.

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Let $\Lambda : T \to \mathcal{F}$ be a correspondence. How do we guarantee that

$$\min\left\{\int_S \phi(s) d\lambda(s) : \lambda \in \Lambda(t')\right\} \geqslant \min\left\{\int_S \phi(s) d\lambda(s) : \lambda \in \Lambda(t)\right\}$$

for all increasing functions $\phi : S \to \mathbb{R}$ and $t' > t$?

First order stochastic dominance: the MEU case

Definition: Let $\Lambda : T \to \mathcal{F}$ be a correspondence.

Λ is FSD-increasing if, for all $t' > t$, the following holds:

for all $\lambda' \in \Lambda(t')$ there is $\lambda \in \Lambda(t)$ such that $\lambda' \gtrsim_{FSD} \lambda$.

Theorem: The function

$$\Phi(t) = \min \left\{ \int_S \phi(s) d\lambda(s) : \lambda \in \Lambda(t) \right\}$$

is increasing in t for increasing functions $\phi : S \to \mathbb{R}$ if and only if $\Lambda : T \to \mathbb{R}$ is increasing.

(Necessity part requires Λ to be convex-valued.)

Example: $\Lambda(t') \subseteq \Lambda(t)$ whenever $t' > t$.

FSD for comparative statics: the EU case

An agent chooses action $x \in X \subset \mathbb{R}$ under uncertainty to maximize

$$v(x,t) = \int_S u(x,s) d\lambda(s,t)$$

u is supermodular if $u(x'', s) - u(x', s)$ is increasing in s for all $x'' > x'$.

Basic result 2: The function v is supermodular in (x,t) if

(i) $u(x,s)$ is supermodular and

(ii) $\lambda(\cdot, t') \succeq_{FSD} \lambda(\cdot, t)$ if $t' > t$.

Interpretation: the supermodularity of u guarantees that $\arg\max_{x \in X} u(x,s)$ is increasing in s (Milgrom-Shannon Theorem).

If λ is FSD-increasing, then $\arg\max_{x \in X} v(x,t)$ is increasing in t.

Changing stochastic environments

Proof: $\Delta(t) := v(x'', t) - v(x', t) = \int [u(x'', s) - u(x', s)] d\lambda(s, t)$.

If $x'' > x'$, then $\delta(s) = u(x'', s) - u(x', s)$ is increasing in s.

So Δ is increasing in t if λ is FSD-increasing. QED

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Example: An agent lives for two periods.
Income today is w_1 and tomorrow's income s is stochastic.

The expected utility of saving $x \in [0, w_1]$ is

$$U(x, t) = \int_S [u_1(w_1 - x) + \beta u_2(Rx + s)] \, d\lambda(s, t).$$

If u_2 is concave, $(x, s) \to u_1(w_1 - x) + \beta u_2(Rx + s)$ is submodular .

Assuming this, if λ is FSD-increasing, then U is submodular

and $\arg\max_x U(x, t)$ decreases with t.

FSD for comparative statics: the MEU case

If the agent is ambiguity averse, his objective function is

$$v(x,t) = \min \left\{ \int_S u(x,s) d\lambda(s) : \lambda \in \Lambda(t) \right\}.$$

What set-generalization of an FSD shift will guarantee the supermodularity of v?

The property on Λ needed for comparative statics is different from the one needed to compare utilities.

FSD for comparative statics: the MEU case

A possible condition: $\Lambda(t')$ dominates $\Lambda(t)$ if every distribution in $\Lambda(t')$ dominates every distribution $\Lambda(t)$.

Choose $x' > x$ and suppose

$v(x,t) = \int u(x,s)d\hat{\lambda}(s)$ for some $\hat{\lambda} \in \Lambda(t)$ and

$v(x',t') = \int u(x',s)d\tilde{\lambda}(s)$ for some $\tilde{\lambda} \in \Lambda(t')$.

Since $\tilde{\lambda} \gtrsim_{FSD} \hat{\lambda}$ and $x' > x$, we obtain

$$\begin{aligned} v(x,t) + v(x',t') &= \int u(x,s)d\hat{\lambda}(s) + \int u(x',s)d\tilde{\lambda}(s) \\ &\geqslant \int u(x,s)d\tilde{\lambda}(s) + \int u(x',s)d\hat{\lambda}(s) \\ &\geqslant v(x,t') + v(x',t). \end{aligned}$$

Lattices and Supermodularity

Definition: A partially ordered set $(X, \geqslant_X)$ is a lattice if every two elements has a least upper bound (supremum) and a greatest lower bound (infimum).

We denote the supremum of x and y by $x \vee y$ and their infimum by $x \wedge y$.

Example 1: $(\mathbb{R}^\ell, \geqslant)$ is a lattice, where $\geqslant$ is the product order, i.e. $x \geqslant y$ if $x_i \geqslant y_i$ for $i = 1, 2, .., \ell$. Indeed,

$$\begin{aligned} x \vee y &= (\max\{x_1, y_1\}, \max\{x_2, y_2\}, ..., \max\{x_\ell, y_\ell\}) \\ x \wedge y &= (\min\{x_1, y_1\}, \min\{x_2, y_2\}, ..., \min\{x_\ell, y_\ell\}). \end{aligned}$$

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Example 2: Distributions on $S \subset \mathbb{R}$ is a lattice when ordered by first order stochastic dominance.

$$\begin{aligned} (\lambda \vee \lambda')(s) &= \min\{\lambda(s), \lambda'(s)\} \\ (\lambda \wedge \lambda')(s) &= \max\{\lambda(s), \lambda'(s)\} \end{aligned}$$

Preliminary definitions

Let $(X, \geqslant_X)$ be a lattice.

A function $f : (X, \geqslant_X) \rightarrow \mathbb{R}$ is a supermodular function if, for any x, x' in X,

$$f(x \wedge x') + f(x \vee x') \geqslant f(x) + f(x').$$

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For $f : (\mathbb{R}^\ell_+, \geqslant) \to \mathbb{R}$, supermodularity is equivalent to the following:

for any $i \in \{1, 2, ..., \ell\}$, with $x_i'' > x_i'$,

$$f(x_i'', x_{-i}) - f(x_i', x_{-i}) \text{ is increasing in } x_{-i}.$$

If f is a production function, this says that the marginal productivity of factor i increases as the input of other factors, x_{-i} is raised.

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If f is differentiable, the supermodularity of f is equivalent to

$$\frac{\partial^2 f}{\partial x_i \partial x_j}(x) \geqslant 0 \text{ for all } x, \text{ and } i \neq j.$$

Supermodular production functions and comparative statics

The significance of supermodular production functions comes from the following

Basic result 3: Suppose the production function $f : (\mathbb{R}^{\ell}_{+}, \geqslant) \to \mathbb{R}$ is supermodular and the firm chooses inputs to maximize its profit

$$\Pi(x; p) = f(x) - p \cdot x,$$

where $p = (p_1, p_2, ..., p_\ell) \gg 0$ are the input prices.

Then the demand for factors displays complementarity in this sense:

a fall in the price of factor i raises the demand for *all* factors.

Multi-output production

Suppose the firm produces multiple outputs.

Associated with input vector $x \in \mathbb{R}^\ell_+$ is the *set* of output vectors $\Gamma(x) \subset \mathbb{R}^J_+$.

In other words, the firm can produce with x any output vector

$$y = (y_1, y_2, ..., y_J) \in \Gamma(x).$$

Let $q = (q_1, ..., q_J) \gg 0$ be the vector of output prices. Then the firm chooses x and $y \in \Gamma(x)$ to maximize its profit $q \cdot y - p \cdot x$.

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We define the revenue function

$$f(x) = \max\{q \cdot y : y \in \Gamma(x)\}.$$

Equivalently, the firm

$$\text{chooses } x \in \mathbb{R}_+^\ell \text{ to maximize } \Pi(x, p) = f(x) - p \cdot x.$$

There is complementarity in factor demand if f is supermodular.

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$$\text{chooses } x \in \mathbb{R}^\ell_+ \text{ to maximize } \Pi(x, p) = f(x) - p \cdot x.$$

There is complementarity in factor demand if f is supermodular.

What conditions on Γ will guarantee that f is supermodular?

Supermodular correspondences

Definition:

$(X, \geqslant_X)$ is a lattice and $(Y, \geqslant_Y)$ a partially ordered real vector space.

A correspondence $\Gamma : X \to Y$ is upper supermodular if for any $y \in \Gamma(x)$ and $y' \in \Gamma(x')$ there is $z \in \Gamma(x \wedge x')$ and $z' \in \Gamma(x \vee x')$ such that

$$z + z' \;\geqslant_Y\; y + y'.$$

Equivalently,

$$\frac{1}{2}z + \frac{1}{2}z' \;\geqslant_Y\; \frac{1}{2}y + \frac{1}{2}y'.$$

The correspondence Γ is lower supermodular if for any $z \in \Gamma(x \wedge x')$ and $z' \in \Gamma(x \vee x')$ there is $y \in \Gamma(x)$ and $y' \in \Gamma(x')$ that satisfy the above inequality.

This is a twofold generalization of the standard notion of supermodularity.

Supermodular correspondences

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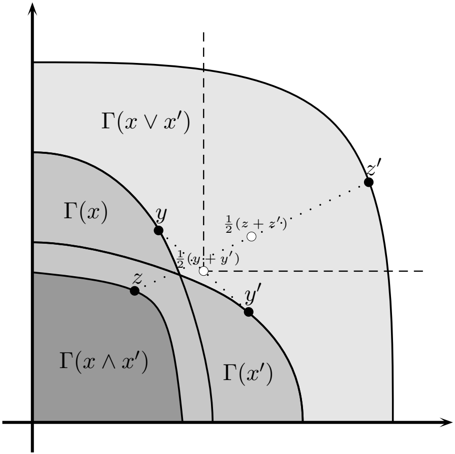
$$z + z' \geqslant_Y y + y'.$$



The main result

Recall that X is a lattice, while Y is a partially ordered real vector space.

Main Theorem: For any positive linear functional $\phi : Y \to \mathbb{R}$,

(i) if correspondence $\Gamma : X \to Y$ is upper supermodular then function $f : X \to \mathbb{R}$, given by

$$f(x) = \sup\left\{\phi(y) : y \in \Gamma(x)\right\},$$

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is supermodular.

Necessity: Whenever values of Γ are compact and convex subsets of a Euclidean space, then the converse is also true.

The main result

Proof: By upper supermodularity of Γ, for any $y \in \Gamma(x)$ and $y' \in \Gamma(x')$ there is some $z \in \Gamma(x \wedge x')$ and $z' \in \Gamma(x \vee x')$ such that

$$z + z' \geqslant_Y y + y'.$$

For any positive linear functional $\phi : Y \to \mathbb{R}$,

$$\begin{aligned}
\phi(y) + \phi(y') &= \phi\big(y + y'\big) \\
&\leqslant \phi\big(z + z'\big) \\
&= \phi(z) + \phi(z') \\
&\leqslant \sup\big\{\phi(y) : y \in \Gamma(x \wedge x')\big\} + \sup\big\{\phi(y) : y \in \Gamma(x \vee x')\big\} \\
&= f(x \wedge x') + f(x \vee x').
\end{aligned}$$

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Proof: By upper supermodularity of Γ, for any $y \in \Gamma(x)$ and $y' \in \Gamma(x')$ there is some $z \in \Gamma(x \wedge x')$ and $z' \in \Gamma(x \vee x')$ such that

$$z + z' \ \geqslant_Y \ y + y'.$$

For any positive linear functional $\phi : Y \to \mathbb{R}$,

$$\begin{aligned} \phi(y) + \phi(y') &= \phi(y + y') \\ &\leqslant \phi(z + z') \\ &= \phi(z) + \phi(z') \\ &\leqslant \sup\{\phi(y) : y \in \Gamma(x \wedge x')\} + \sup\{\phi(y) : y \in \Gamma(x \vee x')\} \\ &= f(x \wedge x') + f(x \vee x'). \end{aligned}$$

By taking the supremum over the left hand side of the inequality, we conclude that $f(x) + f(x') \leqslant f(x \wedge x') + f(x \vee x')$.

Hence the function f is supermodular. QED

Application 1: multi-output production

For a firm with the production correspondence $\Gamma : \mathbb{R}^I_+ \to \mathbb{R}^J_+$,

factor demand complementarity holds if the revenue function

$$f(x) = \max\{q \cdot y : y \in \Gamma(x)\}$$

is supermodular.

By the Main Theorem (i), f is supermodular if Γ is upper supermodular, i.e.,

for any $y \in \Gamma(x)$ and $y' \in \Gamma(x')$ there is $z \in \Gamma(x \wedge x')$ and $z' \in \Gamma(x \vee x')$ such that

$$z + z' \geqslant y + y'.$$

Upper supermodular production correspondences

Example 1: The correspondence $\Gamma : \mathbb{R}^2_+ \to \mathbb{R}^2_+$, where

$$\Gamma(x_1, x_2) = \left\{(y_1, y_2) \in \mathbb{R}^2_+ : \sqrt{y_1^2 + y_2^2} \leqslant f(x_1, x_2)\right\},$$

so long as $f : \mathbb{R}^2_+ \to \mathbb{R}_+$ is a supermodular function.

Example 2: More generally

$$\Gamma(x) = f(x)Z := \{y \in \mathbb{R}^J_+ : y = zf(x) \text{ for some } z \in Z\}$$

where $Z \subset \mathbb{R}^J_+$ is convex and comprehensive, i.e. if $z \in Z$ then $\tilde{z} \in Z$ if $0 \leqslant \tilde{z} \leqslant z$.

Upper supermodular production correspondences

Example 3: The correspondence $\Gamma : \mathbb{R}^3_+ \to \mathbb{R}^2_+$, where

$$\Gamma(x_1, x_2, x_3) := \big\{(y_1, y_2) \in \mathbb{R}^2_+ : y_1 \leqslant x_1 \cdot x_2 \cdot z, \\ \text{and} \quad y_2 \leqslant \sqrt{x_1} + \sqrt{x_3 - z}, \text{ where } z \in [0, x_3]\big\}.$$

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Example 4: The correspondence $\Gamma : \mathbb{R}^l_+ \to \mathbb{R}^n$, where

$$\Gamma(x) = \{f(x, z) : (x, z) \in M\}$$

and $f : M \to \mathbb{R}^n$ is a supermodular function.

In Example 3, $f(x_1, x_2, x_3, z) = \left(x_1 x_2 z\, ,\, \sqrt{x_1} + \sqrt{x_3 - z}\right)$ is a supermodular function.

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Example 5: The sum of any two upper supermodular correspondences is upper supermodular.

Application 2: maxmin expected utility

What assumption on $\Lambda : T \to \mathcal{F}$ will guarantee the supermodularity of

$$v(x,t) = \min\left\{\int_S u(x,s)d\lambda(s) : \lambda \in \Lambda(t)\right\}?$$

Definition: $\Lambda : T \to \mathbb{R}$ is strongly FSD-increasing if, for $t' \geqslant t$, $\lambda' \in \Lambda(t')$, and $\lambda \in \Lambda(t)$, there is some $\mu' \in \Lambda(t')$ and $\mu \in \Lambda(t)$ such that

$$\lambda' \succeq_{FSD} \mu,\ \mu' \succeq_{FSD} \lambda, \text{ and } \tfrac{1}{2}\lambda' + \tfrac{1}{2}\lambda = \tfrac{1}{2}\mu' + \tfrac{1}{2}\mu.$$

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$$\lambda' \succeq_{FSD} \mu,\ \mu' \succeq_{FSD} \lambda, \text{ and } \tfrac{1}{2}\lambda' + \tfrac{1}{2}\lambda = \tfrac{1}{2}\mu' + \tfrac{1}{2}\mu.$$

Example 1: If, for any $\lambda \in \Lambda(t)$ and $\lambda' \in \Lambda(t')$,

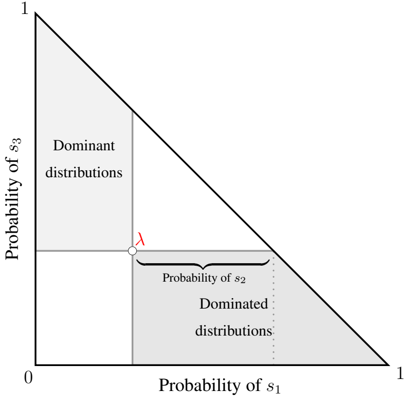
$$\lambda \vee \lambda' \in \Lambda(t') \text{ and } \lambda \wedge \lambda' \in \Lambda(t).$$

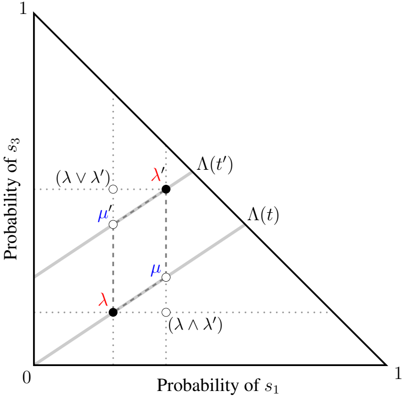
Specific instance: $\Lambda(t) = [\underline{\theta}(t), \bar{\theta}(t)]$ where $\bar{\theta}(t) \succeq_{FSD} \underline{\theta}(t)$ and both $\bar{\theta}$ and $\underline{\theta}$ are FSD-increasing.

Application 2: maxmin expected utility

Example 2: $\Lambda(t)$ = All distributions on $\mathcal{S}$ with mean t.

Illustration when $s_1 < s_2 < s_3$.





Application 2: maxmin expected utility

Theorem: Let X and T be subsets of $\mathbb{R}$. The function $v : X \times T \to \mathbb{R}$ given by

$$v(x,t) = \min\left\{\int_S u(x,s)d\lambda(s) : \lambda \in \Lambda(t)\right\},$$

is supermodular in (x,t) if

(i) $u(x,s)$ is supermodular and

(ii) Λ is strongly FSD-increasing.

If Λ is compact and convex-valued, then v is supermodular for all supermodular u only if Λ is strongly FSD-increasing.

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If Λ is compact and convex-valued, then v is supermodular for all supermodular u only if Λ is strongly FSD-increasing.

Note: If v is supermodular, $\arg\max_{x \in X} v(x,t)$ is increasing in t.

Application 2: maxmin expected utility

Proof of sufficiency: Let $S := \{s_i\}_{i=1}^{\ell+1}$ such that $s_1 < \ldots < s_{\ell+1}$.

For a given distribution $\lambda \in \Lambda(t)$,

$$\int u(x,s)d\lambda(s) = u(x,s_{\ell+1}) + \sum_{i=1}^{\ell} \left[u(x,s_i) - u(x,s_{i+1})\right]\lambda(s_i).$$

Therefore, $v(x,t) = u(x,s_{\ell+1}) + \min\left\{\sum_{i=1}^{\ell} y_i : y \in \Gamma(x,t)\right\}$,

where the correspondence $\Gamma : X \times T \to \mathbb{R}^\ell$ is defined by

$$\Gamma(x,t) = \left\{y \in \mathbb{R}^\ell : y_i = \left[u(x,s_i) - u(x,s_{i+1})\right]\lambda(s_i) \text{ for some } \lambda \in \Lambda(t)\right\}.$$

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By Main Theorem (ii),

v is supermodular if (and essentially, only if) Γ is lower supermodular.

Γ is lower supermodular if u is supermodular and Λ is strongly FSD-increasing.

Application 2: maxmin expected utility

$$\Gamma(x,t) = \Big\{ y \in \mathbb{R}^\ell : y_i = \big[u(x,s_i) - u(x,s_{i+1})\big]\lambda(s_i) \text{ for some } \lambda \in \Lambda(t) \Big\}.$$

Take any $x' \geqslant x$ in X, $t' \geqslant t$ in T, and let $z \in \Gamma(x,t)$ and $z' \in \Gamma(x',t')$.

By definition, there are distributions $\lambda \in \Lambda(t)$ and $\lambda' \in \Lambda(t')$ such that

$z_i = \delta_i(x)\lambda(s_i)$ and $z'_i = \delta_i(x')\lambda'(s_i)$, where $\delta_i(x) = u(x,s_i) - u(x,s_{i+1})$.

Since Λ is strongly FSD-increasing, there is $\mu \in \Lambda(t)$ and $\mu' \in \Lambda(t')$ such that $\mu(s_i) - \lambda'(s_i) = \lambda(s_i) - \mu'(s_i) \geqslant 0$.

And $\delta_i(x') \leqslant \delta_i(x)$ since u is supermodular. Thus,

$$\delta_i(x')\big[(\mu(s_i) - \lambda'(s_i)\big] \;\leqslant\; \delta_i(x)\big[\lambda(s_i) - \mu'(s_i)\big].$$

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Define y and $y' \in \mathbb{R}^n$ by $y_i := \delta_i(x)\mu'(s_i)$ and $y'_i := \delta_i(x')\mu(s_i)$.

So we have shown that $y + y' \leqslant z + z'$.

Since $y \in \Gamma(x,t')$ and $y' \in \Gamma(x',t)$, we conclude that Γ is lower supermodular. QED

Extension to α-maxmin preferences

In fact, the correspondence

$$\Gamma(x,t) = \Big\{ y \in \mathbb{R}^{\ell} : y_i = \big[u(x,s_i) - u(x,s_{i+1})\big]\lambda(s_i) \text{ for some } \lambda \in \Lambda(t) \Big\}.$$

is also *upper* supermodular when u is supermodular and Λ is strongly FSD-increasing (and not just lower supermodular).

Thus by the Main Theorem, both

$$v(x,t) = \min \left\{ \int_S u(x,s) d\lambda(s) : \lambda \in \Lambda(t) \right\}$$

and

$$w(x,t) = \max \left\{ \int_S u(x,s) d\lambda(s) : \lambda \in \Lambda(t) \right\}$$

(and their weighted sums) are supermodular functions.

Application 3: Variational Preferences

The agent's utility from choosing $x \in \mathbb{R}$ is

$$v(x,t) = \min\left\{\int_S u(x,s)d\lambda(s) + c(\lambda,t) : \lambda \in \triangle_S\right\}$$

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Theorem: $v(x,t)$ is supermodular in (x,t) if $u(x,s)$ is supermodular and c satisfies the following condition:

for any $t' \geqslant t$ and distributions λ' and λ, there is μ' and μ such that

$$\lambda' \geq_{FSD} \mu, \;\; \mu' \geq_{FSD} \lambda, \;\; \tfrac{1}{2}\lambda' + \tfrac{1}{2}\lambda = \tfrac{1}{2}\mu' + \tfrac{1}{2}\mu \;\text{ and}$$

$$c(\lambda,t) + c(\lambda',t') \leqslant c(\mu,t) + c(\mu',t').$$

It suffices for c to be submodular in both arguments:

$$c(\lambda,t) + c(\lambda',t') \leqslant c(\lambda \wedge \lambda',t) + c(\lambda \vee \lambda',t').$$

Application 3: Variational Preferences

Proof. Let $S = \{s_i\}_{i=1}^{\ell+1}$. For any $x \in X$ and $\lambda \in \Delta_S$,

$$\int_S u(x,s)d\lambda(s) = u(x, s_{\ell+1}) + \sum_{i=1}^{\ell} \delta_i(x)\lambda(s_i),$$

where $\delta_i(x) := \big[u(x,s_i) - u(x,s_{i+1})\big]$, for $i = 1, \ldots, \ell$. Therefore,

$$v(x,t) = u(x, s_{\ell+1}) + \min\left\{\sum_{i=1}^{\ell} \delta_i(x)\lambda(s_i) + c(\lambda,t) : \lambda \in \Lambda(t)\right\}. \quad (1)$$

When is $\bar{v}(x,t) := \min\left\{\sum_{i=1}^{\ell} \delta_i(x)\lambda(s_i) + c(\lambda,t) : \lambda \in \Lambda(t)\right\}$ spm?

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By the Main Theorem (ii), this holds if the correspondence

$$\bar{\Gamma}(x,t) := \Big\{(\delta_1(x)\lambda(s_1), \delta_2(x)\lambda(s_2) \ldots, \delta_\ell(x)\lambda(s_\ell), c(\lambda,t)) : \lambda \in \triangle_S\Big\},$$

is lower supermodular.

Which holds if $u(x,s)$ is supermodular and $c(\lambda,t)$ is submodular. QED

Application 4: Multiplier preferences

The is the case of variational preferences where

$$c(\lambda, t) := \theta R\big(\lambda \| \lambda^*(\cdot, t)\big),$$

for some $\lambda^*(\cdot, t) \in \triangle_S$, where R is the relative entropy, i.e.,

$$R(\lambda \| \lambda^*(\cdot, t)) := \sum_{s \in S} \pi_s \ln \left(\frac{\pi_s}{\pi_s^*(t)} \right)$$

Note: π_s is the probability of state s in the distribution λ.

$\lambda^*(\cdot, t)$ is the *reference* or *benchmark distribution*.

$R(\lambda \| \lambda^*(\cdot, t)) = 0$ if $\lambda = \lambda^*$ and is positive otherwise.

[Sargent and Hansen (2001), Strzalecki (2011)]

Application 4: Multiplier preferences

Proposition: For any fixed $\lambda^*(\cdot, t)$, the relative entropy

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is a submodular function of $\lambda \in \Delta_S$.

Furthermore, R is a submodular function of (λ, t) if $\lambda^*(\cdot, t)$ is increasing in t with respect to the monotone likelihood ratio order, i.e.,

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Recap: $v(x, t) = \min \left\{ \int_S u(x, s) d\lambda(s) + \theta R(\lambda \| \lambda^*(t)) : \lambda \in \triangle_S \right\}$

is supermodular in (x, t) if

(1) u is supermodular in (x, s) and

(2) λ^* is increasing in t with respect to the monotone likelihood ratio.

Conclusion

We extend the comparative statics of first order stochastic dominance to models with ambiguity aversion.

We extend the comparative statics of production complementarity to models with multiple outputs.

Results are obtained as part of a general theory of supermodular correspondences.