Short Survey of Monotone Comparative Statics Part I March 1st, 2011

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Overview

In today's lecture, I will give a quick survey of the following topics:

1. Comparative statics for one-dimensional optimization problems
2. Supermodular Games
3. Comparative statics for multi-dimensional optimization problems
4. The interval dominance order

The second lecture, on Tuesday, March 8th, will cover the following topics, all of which involve decision making under uncertainty.

5. Comparative statics of optimization problems under uncertainty; related to this, the aggregation of the single crossing property
6. Bayesian games with monotone decision functions; for example, first-price auctions
7. Comparative information and the complete class theorem

One-dimensional comparative statics

Let $X \subseteq \mathbb{R}$ and let $f(\cdot; s) : X \to \mathbb{R}$ be a family of functions parameterized by $s \in S$ (a poset).

We are interested in how $\text{argmax}_{x \in X} f(x; s)$ varies with s.

Standard approach: Assume X is a compact interval and $f(\cdot; s)$ are quasi-concave functions of x. Let $\bar{x}(s)$ be the unique maximizer of $f(\cdot; s)$.

Then $f'(\bar{x}(s^*), s^*) = 0$. Show that $f'(\bar{x}(s^*), s^{**}) \geq 0$ for $s^{**} > s^*$. Then optimum has shifted to the right, i.e., $\bar{x}(s^{**}) \geq \bar{x}(s^*)$.

This approach makes various assumptions, most notably the quasi-concavity of $f(\cdot; s)$. Not the most natural assumption; example:

let x be output, P the inverse demand function, and c the marginal cost of producing good.
The profit function $\Pi(x; c) = xP(x) - cx$ is not naturally concave in x.

One-dimensional comparative statics

Assume that $f(\cdot; s)$ is continuous in $x \in X$ and X is compact. Then argmax$_{x \in X} f(x; s)$ is nonempty. But it need not be singleton or an interval.

First question: how do we compare sets?

Definition: Let S' and S'' be subsets of $\mathbb{R}$. S'' dominates S' in the strong set order ($S'' \geq S'$) if for any for x'' in S'' and x' in S', we have $\max\{x'', x'\}$ in S'' and $\min\{x'', x'\}$ in S'.

Example: $\{3, 5, 6, 7\} \not\geq \{1, 4, 6\}$ but $\{3, 4, 5, 6, 7\} \geq \{1, 3, 4, 5, 6\}$.

Note: if $S'' = \{x''\}$ and $S' = \{x'\}$, then $x'' \geq x'$.

When S'' and/or S' are non-singleton,
largest element in S'' is *larger than* the largest element in S';
smallest element in S'' is *larger than* the smallest element in S'.

One-dimensional comparative statics

Definition: Let S be a poset and $\phi : S \to \mathbb{R}$. Then ϕ has the single crossing property (is a single crossing function) if

$$\phi(s') \geq (>)\, 0 \Longrightarrow \phi(s'') \geq (>)\, 0 \text{ where } s'' > s'.$$

Definition: The family of functions $\{f(\cdot, s)\}_{s\in S}$ obeys single crossing differences if for all $x'' > x'$, the function

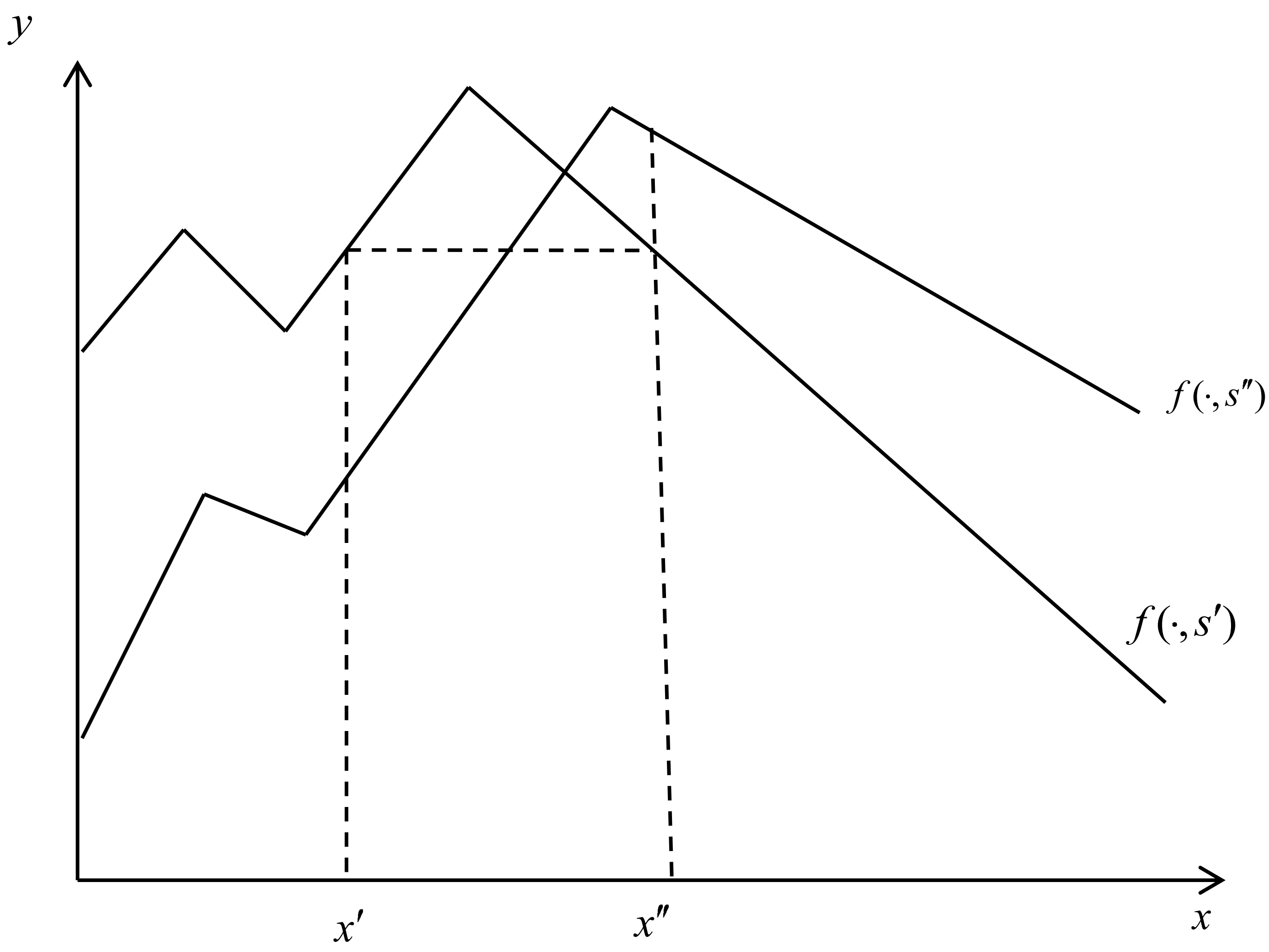
$$\delta(s) = f(x''; s) - f(x'; s) \text{ is a single crossing function.}$$

These are ordinal properties. In particular,

if $\{f(\cdot, s)\}_{s\in S}$ obey single crossing differences, then so does $\{g(\cdot; s)\}_{s\in S}$ where there is a function $H(\cdot; s)$, strictly increasing in x, such that $g(x; s) = H(f(x; s); s)$.

One-dimensional comparative statics

A single crossing differences family:



One-dimensional comparative statics

Definition: The family of functions $\{f(\cdot, s)\}_{s\in S}$ obeys increasing differences if for all $x'' > x'$, the function

$$\delta(s) = f(x''; s) - f(x'; s) \text{ is an increasing function.}$$

Very often, the easiest way to show that a family obeys single crossing differences is to show that some strictly increasing transformation of this family has increasing differences.

Proposition 1: Let S be an open subset of $\mathbb{R}^l$ and X an open interval. Then a sufficient (and necessary) condition for the family $\{f(\cdot, s)\}_{s\in S}$ to obey increasing differences is that

$$\frac{\partial^2 f}{\partial x \partial s_i}(x, s) \geq 0$$

at every point (x, s) and for all i.

One-dimensional comparative statics

Theorem 1: (Milgrom-Shannon)[a] The family $\{f(\cdot;s)\}_{s\in S}$ obeys single crossing differences if and only if $\text{argmax}_{x\in Y} f(x;s)$ is increasing in s for all $Y \subseteq X$.

Proof: Assume $s'' > s'$ and $x'' \in \text{argmax}_{x\in Y} f(x;s'')$, and $x' \in \text{argmax}_{x\in Y} f(x;s')$. We have to show that $\max\{x',x''\} \in \text{argmax}_{x\in Y} f(x;s'')$ and $\min\{x',x''\} \in \text{argmax}_{x\in Y} f(x;s')$.
We need only consider the case where $x' > x''$.

Since $x' \in \text{argmax}_{x\in Y} f(x;s')$, we have $f(x';s') \geq f(x'';s')$. By single crossing differences, $f(x';s'') \geq f(x'';s'')$ so $x' \in \text{argmax}_{x\in Y} f(x;s'')$.

Furthermore, $f(x';s') = f(x'';s')$ so that $x'' \in \text{argmax}_{x\in Y} f(x;s')$. If not, $f(x';s') > f(x'';s')$ which implies (by single crossing differences) that $f(x';s'') > f(x'';s'')$, contradicting the assumption that $f(\cdot;s'')$ is maximized at x''.

Necessity: follows from definition of single crossing differences! **QED**

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[a] Detailed references are on the last slide.

One-dimensional comparative statics

Application 1: Let $\Pi(x; -c) = xP(x) - cx$. Then $\{\Pi(\cdot, -c)\}_{-c \in \mathbb{R}_-}$ obey increasing differences, since

$$\frac{\partial^2 \Pi}{\partial x \partial c} = -1.$$

By MCS Theorem, $\text{argmax}_{x \in X} \Pi(x, -c)$ is increasing in $-c$.

In other words, the profit-maximizing output decreases as the marginal cost of output increases.

Application 2: Let $\Pi(q; N) = NqP(q) - C(Nq)$. Make an increasing transformation and consider

$$\tilde{\Pi}(q; N) = qP(q) - \frac{C(Nq)}{N}.$$

The family $\{\tilde{\Pi}(\cdot; N)\}_{n \in N}$ has increasing differences if $C'' < 0$ and decreasing differences if $C'' > 0$.

So $\text{argmax}_{q > 0} \Pi(q; N)$ is increasing (decreasing) with N if marginal cost is decreasing (increasing) with output.

One-dimensional comparative statics

Application 3: Bertrand Oligopoly with differentiated products, with

$$\begin{aligned}\Pi_a(p_a, p_{-a}) &= (p_a - c_a)D_a(p_a, p_{-a}) \\ \ln \Pi_a(p_a, p_{-a}) &= \ln(p_a - c_a) + \ln D_a(p_a, p_{-a})\end{aligned}$$

So $\{\Pi_a(\cdot, p_{-a})\}_{-a\in -A}$ has single crossing differences if $\{\ln \Pi_a(\cdot, p_{-a})\}_{-a\in -A}$ has increasing differences. We require

$$\frac{\partial^2}{\partial p_a \partial p_{-a}}[\ln \Pi_a] \geq 0. \text{ Equivalently,}$$

$$\frac{\partial}{\partial p_{-a}}\left[-\frac{p_a}{D_a}\frac{\partial D_a}{\partial p_a}\right] \leq 0;$$

i.e., firm a's own-price elasticity of demand decreases with p_{-a}. If this assumption holds, $\text{argmax}_{p_a \in P} \Pi_a(p_a, p_{-a})$ increases with p_{-a}.

In other words, firm a's optimal price is *increasing* in the price charged by other firms. Firms' strategies are complements.

Supermodular Games

The Bertrand game is an example of a supermodular game.

A supermodular game is one where the best response of each agent is increasing with the strategies of the other agents.

In the next few slides we take a brief look at the properties of these games.

We do not assume that the payoff functions of the agents in the game are quasiconcave. Therefore, the best response map need not be a convex-valued, upper hemi-continuous correspondence. For this reason, the ‘standard’ proof of equilibrium existence via Kakutani’s fixed point theorem cannot be applied.

Instead, we appeal to the monotonicity of the best response map and use another fixed point theorem – Tarski’s.

Supermodular Games

Let $X = \Pi_{i=1}^{N} X_i$, where each X_i is a compact interval of $\mathbb{R}$.

Theorem 2: (Tarski) Suppose $\phi : X \to X$ is an increasing function. Then ϕ has a fixed point. In fact,

$$x^{**} = \sup\{x \in X : x \leq \phi(x)\}$$

is a fixed point and is the *largest* fixed point, i.e., for any other fixed point x^*, we have $x^* \leq x^{**}$.

Note: ϕ need not be continuous.

Corollary 1: Suppose $\phi(\cdot, t) : X \to X$ is increasing in (x, t). Then the largest fixed point of $\phi(\cdot, t)$ is increasing in t.

Supermodular Games

Bertrand Oligopoly:

Assume the set of firms is A; the typical firm a chooses its price from the compact interval P to maximize $\Pi_a(p_a, p_{-a}) = (p_a - c_a)D_a(p_a, p_{-a})$.

Recall: if own-price elasticity is decreasing in p_{-a} then $\{\Pi_a(\cdot, p_{-a})\}_{p_{-a} \in P_{-a}}$ obeys single crossing differences.

Consequently, firm a's best response set

$$B_a(p_{-a}) = \text{argmax}_{p_a \in P} \Pi_a(p_a, p_{-a}) \text{ is increasing in } p_{-a}.$$

Define $\bar{B}_a(p_{-a}) = \max \left[\text{argmax}_{p_a \in P} \Pi_a(p_a, p_{-a})\right]$;
this is the *largest* best response to p_{-a}.

$\bar{B}_a$ is an increasing function of p_{-a}.

Supermodular Games

Define $\bar{P} = P \times P \times ... \times P$ and the map $\bar{B} : \bar{P} \to \bar{P}$ by

$$\bar{B}(p) = \left(\bar{B}_a(p_{-a})\right)_{a \in A} .$$

A fixed point of this map is a NE of the game.

Since $\bar{B}$ is an increasing function, Tarski's Fixed Point Theorem guarantees that a fixed point exists.

Specifically,

$$p^* = \sup\{p \in \bar{P} : p \leq \bar{B}(p)\}$$

is a fixed point of the map $\bar{B}$ and thus a pure strategy Nash equilibrium of the game.

In fact, this is the *largest* NE, i.e., suppose $\hat{p}$ is another NE; then $p^* > \hat{p}$.

We can do comparative statics exercises on the largest NE...

Supermodular Games

What happens to the largest NE when firm $\tilde{a}$ experiences an increase in marginal cost from $c_{\tilde{a}}$ to $c'_{\tilde{a}}$? Recall

$$\ln \Pi_{\tilde{a}}(p_{\tilde{a}}, p_{-\tilde{a}}, c_{\tilde{a}}) = \ln(p_{\tilde{a}} - c_{\tilde{a}}) + \ln D_{\tilde{a}}(p_{\tilde{a}}, p_{-\tilde{a}}).$$

Observe that

$$\frac{\partial}{\partial p_{\tilde{a}} \partial c_{\tilde{a}}} [\ln \Pi_{\tilde{a}}] > 0.$$

By the MCS Theorem, firm a's best response increase with $c_{\tilde{a}}$ (for fixed p_{-a}). Formally,

$$B_{\tilde{a}}(p_{-\tilde{a}}, c'_{\tilde{a}}) \geq B_{\tilde{a}}(p_{-\tilde{a}}, c_{\tilde{a}}).$$

This implies that $\bar{B}(p, c'_{\tilde{a}}) \geq \bar{B}(p, c_{\tilde{a}})$. So largest fixed point of $\bar{B}(\cdot, c'_{\tilde{a}})$ is larger than the largest fixed point of $\bar{B}(\cdot, c_{\tilde{a}})$ (by Corollary 1).

In other words, if firm $\tilde{a}$'s marginal cost increases from $c_{\tilde{a}}$ to $c'_{\tilde{a}}$, the largest NE increases: *every firm increases its price*.

Multidimensional comparative statics

Definition: A partially ordered set $(X, \geq)$ is a lattice if for every x and x' in X, the set $\{x, x'\}$ has an infimum and a supremum.

We denote the supremum by $x' \vee x''$ and their infimum by $x' \wedge x''$.

For example, $X = \mathbb{R}^l$ with the usual product order is a lattice.
In that case,

$x' \vee x'' = (\max\{x'_1, x''_1, \max\{x'_2, x''_2\}, ..., \max\{x'_l, x''_l\})$ and

$x' \wedge x'' = (\min\{x'_1, x''_1, \min\{x'_2, x''_2\}, ..., \min\{x'_l, x''_l\})$.

Multidimensional comparative statics

For comparative statics, we need to develop a way of ordering subsets of a lattice.

Definition: Let $(X, \geq)$ be a lattice and S' and S'' subsets of X. S'' dominates S' in the strong set order ($S'' \geq S'$) if for any for x'' in S'' and x' in S', we have $x' \vee x''$ in S'' and $x' \wedge x''$ in S'.

(1) If $S'' = \{x''\}$ and $S' = \{x'\}$, then $S'' \geq S'$ if and only if $x'' \geq x'$.

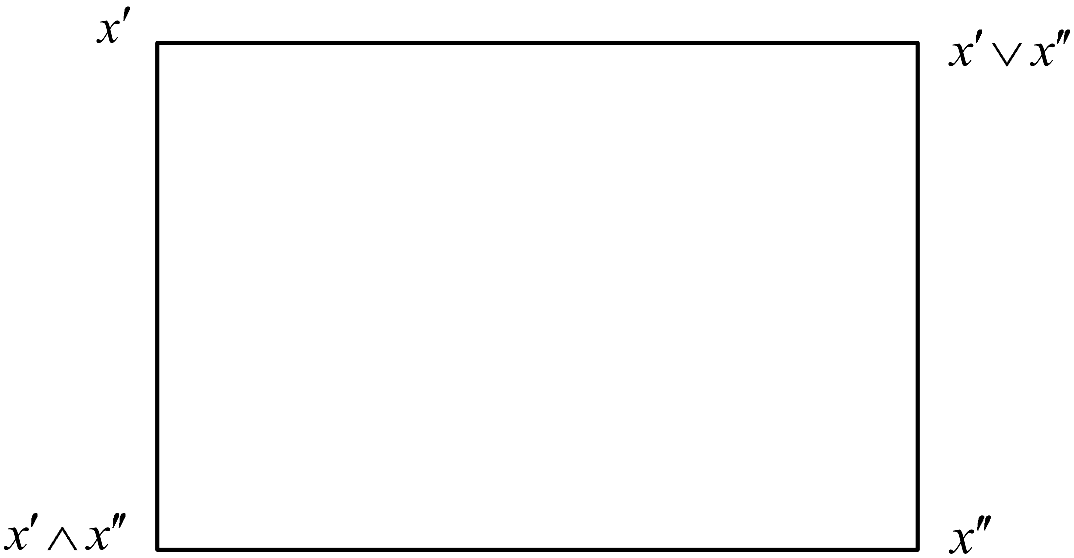
(2) Suppose S' and S'' both contain their suprema; then $S'' \geq S'$ implies that $\sup S'' \geq \sup S'$.

(3) Suppose S' and S'' both contain their infima; then $S'' \geq S'$ implies that $\inf S'' \geq \inf S'$.

Multidimensional comparative statics

Definition: Let X be a sublattice of $\mathbb{R}^l$ and $f : X \to \mathbb{R}$. The function f is supermodular (SPM) if

$$f(x' \vee x'') - f(x'') \geq f(x') - f(x' \wedge x'').$$



Definition: The function f is quasisupermodular (QSM) if

$$f(x') - f(x' \wedge x'') \geq \ (>) \, 0 \implies f(x' \vee x'') - f(x'') \geq \ (>) \, 0.$$

Multidimensional comparative statics

Proposition 2: For the lattice $(\mathbb{R}^l, \geq)$, a C^2 function $f : \mathbb{R}^l \to \mathbb{R}$ is supermodular if and only if

$$\frac{\partial^2 f}{\partial x_i \partial x_j}(x) \geq 0 \text{ at every point } x \in \mathbb{R}^l.$$

Recall: $\{f(\cdot; s)\}_{s \in S}$ obeys single crossing differences if, for any $x'' > x'$, the function

$$\delta(s) = f(x''; s) - f(x'; s)$$

is a single crossing function.

Theorem 3: (Milgrom and Shannon) Let $(X, \geq)$ be a lattice and $\{f(\cdot; s)\}_{s \in S}$ a family of functions obeying single crossing differences, with $f(\cdot; s)$ a quasisupermodular function of x (for each s). Then $\text{argmax}_{x \in X} f(x; s)$ is increasing in s.

Multidimensional comparative statics

Application: Let x denote the vector of inputs (drawn from $X = \mathbb{R}^l_+$), $p = (p_1, p_2, ..., p_l)$ the vector of input prices, and V the revenue function mapping input vector x to revenue (in $\mathbb{R}$). The firm's profit is

$$\Pi(x; p) = V(x) - p \cdot x.$$

Note that

$$\frac{\partial^2 \Pi}{\partial x_i \partial p_j}(x; p) = -1,$$

so there is decreasing differences. Suppose

$$\frac{\partial^2 V}{\partial x_i \partial x_j}(x; p) \geq 0 \text{ for all } x > 0,$$

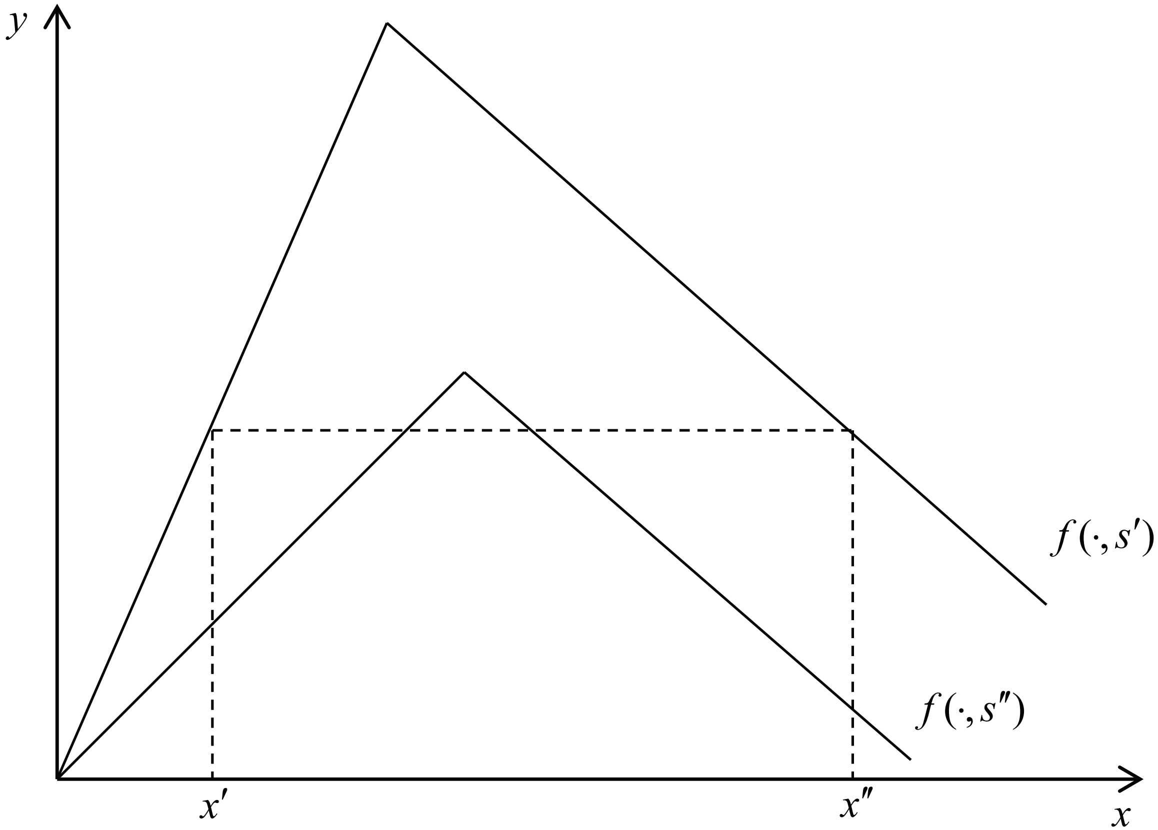
then Π is supermodular (because $\frac{\partial^2 \Pi}{\partial x_i \partial x_j}(x; p) \geq 0$). By Theorem 3,

$$\text{argmax}_{x \in X} \Pi(x; p') \geq \text{argmax}_{x \in X} \Pi(x; p'') \text{ if } p' < p''.$$

A fall in the price of one factor raises demand for *all* factors.

The interval dominance order

Single crossing differences is not a panacea...



Single crossing differences does *not* hold above – even though this is just a family of concave functions ordered by their peaks.

The interval dominance order

Let $X \subseteq \mathbb{R}$. The set $Y \subseteq X$ is an interval of X if, whenever x^* and x^{**} are in X, then any $x \in X$ such that $x^* < x < x^{**}$ is also in Y.

Notation: $[x^*, x^{**}] = \{x \in X : x^* \leq x \leq x^{**}\}$.

Definition: The family $\{f(\cdot; s)\}_{s \in S}$ obeys single crossing differences if for any $x'' > x'$ and $s'' > s'$,

$$f(x''; s') - f(x'; s') \geq (>)\, 0 \implies f(x''; s'') - f(x'; s'') \geq (>)\, 0.$$

Definition: The family $\{f(\cdot; s)\}_{s \in S}$ obeys the interval dominance order if for any $x'' > x'$ and $s'' > s'$, such that

$f(x''; s') - f(x; s') \geq 0$ for all $x \in [x', x'']$, we have

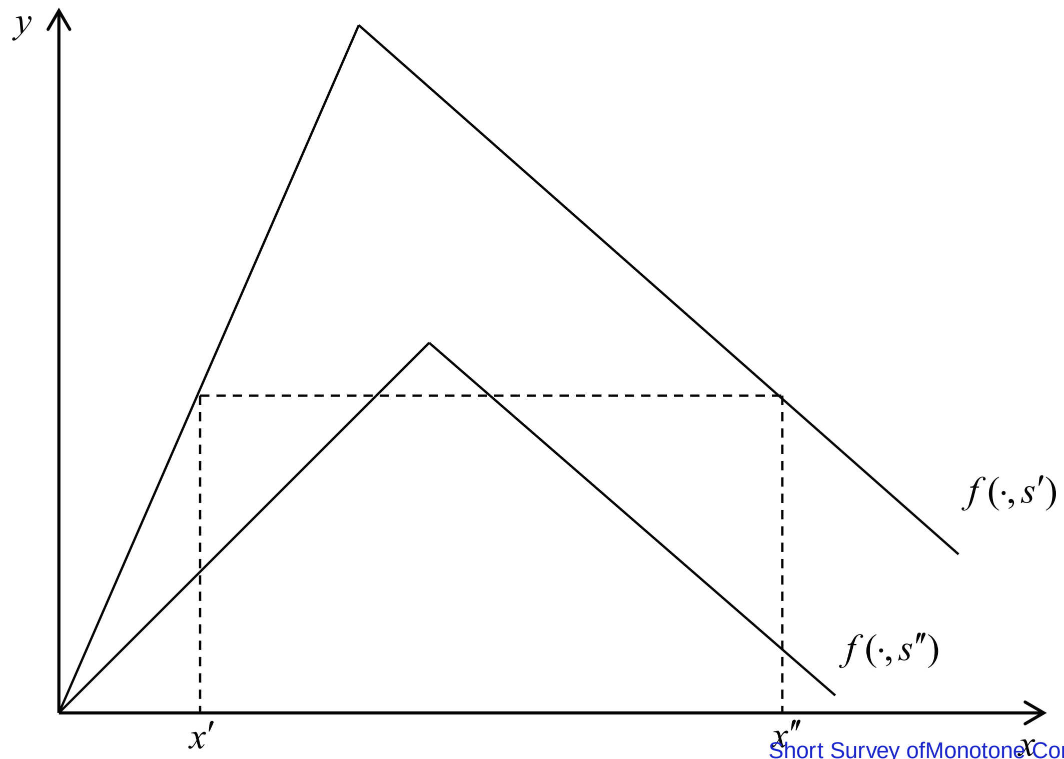
$$f(x''; s') - f(x'; s') \geq (>)\, 0 \implies f(x''; s'') - f(x'; s'') \geq (>)\, 0.$$

The interval dominance order

Definition: The family $\{f(\cdot;s)\}_{s\in S}$ obeys the interval dominance order if for any $x'' > x'$ and $s'' > s'$, such that $f(x'';s') - f(x;s) \geq 0$ for all $x \in [x', x'']$, we have

$$f(x'';s') - f(x';s') \geq (>)\, 0 \implies f(x'';s'') - f(x';s'') \geq (>)\, 0.$$

In diagram below, $\{f(\cdot, s'), f(\cdot, s'')\}$ forms an IDO family.
IDO does *not* require us to compare x' and x'' in this diagram.



The interval dominance order

Consider $X \subset \mathbb{R}$.

Theorem 1: (Milgrom-Shannon) The family $\{f(\cdot; s)\}_{s \in S}$ obeys single crossing differences if and only if $\text{argmax}_{x \in Y} f(x; s)$ is increasing in s for all $Y \subseteq X$.

Theorem 4: (Quah-Strulovici) The family $\{f(\cdot; s)\}_{s \in S}$ obeys the interval dominance order if and only if $\text{argmax}_{x \in Y} f(x; s)$ is increasing in s for all *intervals* $Y \subseteq X$.

Theorem 4 is useful only if there are simple ways of checking that a family obeys the interval dominance order. One such condition is given by Proposition 3 in the next slide.

The interval dominance order

Consider the case where X is an *interval of* $\mathbb{R}$*.*

A simple sufficient condition for $\{f(\cdot; s)\}_{s \in S}$ to obey single crossing differences is the following:

for any $\bar{s} > s$, there is scalar $k > 0$ such that $f'(x; \bar{s}) \geq k f'(x; s)$ for all $x \in X$.

This is clear, since the condition guarantees that, for any $x^{**} > x^*$,

$$f(x^{**}; \bar{s}) - f(x^*; \bar{s}) \geq k \left[f(x^{**}; s) - f(x^*; s) \right].$$

Proposition 3: Let X be an interval of $\mathbb{R}$ and let $\{f(\cdot; s)\}_{s \in S}$ be family of real-valued functions with the following property: for any $\bar{s} > s$, there is a nondecreasing positive function $\alpha : X \to \mathbb{R}$ such that

$$f'(x; \bar{s}) \geq \alpha(x) f'(x; s) \text{ for all } x \in X.$$

The interval dominance order

The optimal stopping time problem: at each moment in time, agent gains profit of $\pi(t)$, which can be positive or negative. If agent decides to stop at time x, the present value of his accumulated profit is

$$V(x;r) = \int_0^x e^{-rt}\pi(t)dt$$

where $r > 0$ is the discount rate.

How does optimal stopping time vary with discount rate?

Note that $V'(x;r) = e^{-rx}\pi(x)$. So

(i) there are lots of turning points;
(ii) turning points do not vary with the discount rate.

The interval dominance order

Proposition 4: Suppose

$$V(x;r) = \int_0^x e^{-rt}\pi(t)dt.$$

If $r > \bar{r} > 0$ then $\text{argmax}_{x\geq 0} V(x;\bar{r}) \geq \text{argmax}_{x\geq 0} V(x;r)$.

Proof: We have

$$V'(x;\bar{r}) = e^{-\bar{r}x}\pi(x) = e^{(r-\bar{r})x}V'(x;r).$$

Note that the function $\alpha(x) = e^{(r-\bar{r})x}$ is positive and increasing.

So $\{V(\cdot;r)\}_{r>0}$ obeys the interval dominance order (strictly speaking, with respect to $-r$).[a]

By Theorem 4, $\text{argmax}_{x\geq 0} V(x;\bar{r}) \geq \text{argmax}_{x\geq 0} V(x;r)$. **QED**

[a] It could also be shown that $\{V(\cdot;r)\}_{r>0}$ does *not* obey single crossing differences.

References

Listed below are readings containing the results discussed in the lecture.

Basic monotone comparative statics theorems

Milgrom, P. and C. Shannon (1994): “Monotone Comparative Statics,” *Econometrica*, 62(1), 157-180.

Topkis, D. M. (1998): *Supermodularity and Complementarity.* Princeton: Princeton University Press.

Supermodular games

Milgrom, P. and J. Roberts (1990): “Rationalizability, Learning, and Equilibrium in Games with Strategic Complementarities," *Econometrica,* 58(6), 1255-1277.

Vives, X. (1990): “Nash Equilibrium with Strategic Complementarities," *Journal of Mathematical Economics*, 19, 305-21.

Production complementarities and their implications

Milgrom, P. and J. Roberts (1990): “The economics of modern manufacturing: technology, strategy, and organization," *American Economic Review*, 80(3), 511-528.

References

Interval dominance order

Quah, J. K.-H. and B. Strulovici (2009): “Comparative Statics, Informativeness, and the Interval Dominance Order," *Econometrica,* 77(6), 1949-1992.

Comparative statics of optimal stopping problems (with and without uncertainty)

Quah, J. K.-H. and B. Strulovici (2009): “Comparative Statics, Informativeness, and the Interval Dominance Order," *Econometrica,* 77(6), 1949-1992.

Quah, J. K.-H. and B. Strulovici (2010): “Discounting and Patience in optimal stopping and control problems." http://faculty.wcas.northwestern.edu/ bhs675/Stopping.pdf

Finally, two readings on an important topic we did *not* discuss.

Comparative statics of optimization problems when constraint sets change

Milgrom, P. and C. Shannon (1994): “Monotone Comparative Statics,” *Econometrica*, 62(1), 157-180.

Quah, J. K.-H. (2007): “The Comparative Statics of Constrained Optimization Problems," *Econometrica*, 75(2), 401-431.