Short Survey of Monotone Comparative Statics Part II: Optimization under uncertainty March 8th, 2011

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Overview

In this lecture, I will consider three comparative statics problems related to optimization under uncertainty.

In all three problems my aim is to find conditions under which the single crossing property holds. This can be understood as an *aggregation problem*: if the single crossing property is valid at each realized state, when will it be preserved when one takes expectation?

I will also briefly discuss how different information structures can be compared, using a concept developed by Lehmann.

Single crossing differences

Definition: Let S be a poset and $\phi : S \to \mathbb{R}$. Then ϕ has the single crossing property (is a single crossing function) if

$$\phi(s') \geq (>)\, 0 \Longrightarrow \phi(s'') \geq (>)\, 0 \text{ where } s'' > s'.$$

Definition: The family of functions $\{v(\cdot, s)\}_{s \in S}$ obeys single crossing differences if for all $x'' > x'$, the function

$$\delta(s) = v(x''; s) - v(x'; s) \text{ is a single crossing function.}$$

Theorem: (Milgrom-Shannon) The family $\{v(\cdot; s)\}_{s \in S}$ obeys single crossing differences if and only if $\text{argmax}_{x \in Y} f(x; s)$ is increasing in s for all $Y \subseteq X$.

Now interpret s as the state and assume that the agent chooses the action x *before* the state is realized.

Optimization under uncertainty - Problem I

Then x is chosen to maximize

$$V(x,\theta) = \int_S v(x,s)\lambda(s,\theta)\,ds$$

where $\lambda(\cdot,\theta)$ is the density function over $s \in S \subset \mathbb{R}$ (one-dimensional!).

Given that $\{v(\cdot;s)\}_{s\in S}$ obeys single crossing differences, the action is increasing in the state if the state is known. If the state is uncertain, we would still expect the optimal action to be higher if higher states are more likely.

It suffices that $\{V(\cdot,\theta)\}_{\theta\in\Theta}$ is a single crossing family, i.e.,

$$\Delta(\theta) = V(x'',\theta) - V(x',\theta)$$

is a single crossing function for any $x'' > x'$.

Definition: $\{\lambda(\cdot,\theta)\}_{\theta\in\Theta}$ obeys the monotone likelihood ratio order if

$$\frac{\lambda(s,\theta'')}{\lambda(s,\theta')} \text{ is increasing in } s \text{ whenever } \theta'' > \theta'.$$

Optimization under uncertainty - Problem I

Theorem 1: Let $S \subset \mathbb{R}$ and suppose $\delta : S \to \mathbb{R}$ is a single crossing function and $\{\lambda(\cdot, \theta)\}_{t \in \Theta}$ obeys the MLR order. Then

$$\Delta(\theta) = \int_S \delta(s)\lambda(s, \theta)\, ds \text{ is a single crossing function (of } \theta).$$

Corollary 1: Suppose that $\{v(\cdot; s)\}_{s \in S}$ obeys single crossing differences and $\{\lambda(\cdot, \theta)\}_{\theta \in \Theta}$ obeys the monotone likelihood ratio order. Then $\{V(\cdot; s)\}_{\theta \in \Theta}$ obeys single crossing differences, where

$$V(x, \theta) = \int_S v(x, s)\lambda(s, \theta)\, ds.$$

Consequently, $\text{argmax}_{x \in X} V(x; \theta)$ is increasing in θ.

Proof: Note that

$$\Delta(\theta) = V(x'', \theta) - V(x', \theta) = \int_S [v(x'', s) - v(x', s)]\lambda(s, \theta)\, ds = \int_S \delta(s)\lambda(s, \theta)\, ds$$

Since $\{v(\cdot; s)\}_{s \in S}$ obeys single crossing differences, δ is a single crossing function. Conclusion follows immediately from theorem above. **QED**

Optimization under uncertainty - Problem I

Proof of Theorem 1: Let $\theta'' > \theta'$. We split $\Delta(\theta'') = \int_S \delta(s)\lambda(s,\theta'')\,ds$ into two parts:

$$\Delta(\theta'') = \int_{-\infty}^{s_0} \delta(s)\lambda(s,\theta')\,\frac{\lambda(s,\theta'')}{\lambda(s,\theta')}\,ds + \int_{s_0}^{\infty} \delta(s)\lambda(s,\theta')\,\frac{\lambda(s,\theta'')}{\lambda(s,\theta')}\,ds,$$

where $\delta(s) \leq 0$ for $s < s_0$ and $\delta(s) > 0$ for $s > s_0$. The first term on the right is greater than

$$\frac{\lambda(s_0,\theta'')}{\lambda(s_0,\theta')}\int_{-\infty}^{s_0} \delta(s)\lambda(s,\theta')\,ds$$

while the second term is greater than

$$\frac{\lambda(s_0,\theta'')}{\lambda(s_0,\theta')}\int_{s_0}^{\infty} \delta(s)\lambda(s,t_1)\,ds.$$

Adding up the two lower bounds gives us

$$\Delta(\theta'') \geq \frac{\lambda(s_0,\theta'')}{\lambda(s_0,\theta')}\int_S \delta(s)\lambda(s,\theta')\,ds = \frac{\lambda(s_0,\theta'')}{\lambda(s_0,\theta')}\Delta(\theta').$$

So $\Delta(\theta') \geq (>)\, 0$ implies $\Delta(\theta'') \geq (>)\, 0$. **QED**

Optimization under uncertainty - Problem I

Application: Consider a firm that maximizes profit

$$\Pi(x, -c) = xP(x) - cx.$$

Since $\frac{\partial^2 \Pi}{\partial x \partial} = -1$, the family $\{\Pi(\cdot, -c)\}_{c \in R_+}$ obeys increasing (hence single crossing) differences.

Theorem 1 says that argmax$_{x \geq 0} \Pi(x, -c)$ is increasing in $-c$.

Now suppose that the firm has to choose x *before* c is known. Given its Bernoulli utility function u, the firm's objective function is

$$V(x; t) = \int u(\Pi(x, -c))\lambda(c, \theta)\, dc$$

where $\lambda(\cdot, \theta)$ is a density function (defined over c).
Note that $v(x; -c) \equiv u(\Pi(x, -c))$ obeys single crossing differences.

Theorem 2 says that when higher c becomes more likely (in the MLR sense), then the firm will choose to produce less.

Optimization under uncertainty - Problem II

Problem: Suppose that the firm faces uncertainty, not in its cost, but in its inverse demand function. Formally, it's problem is to choose x to maximize

$$V(x, -c) = \int_T u(xP(x,t) - cx)\lambda(t)\, dt.$$

When is it still true that higher c leads to lower output?

More generally, suppose that for every $t \in T$, the family $\{v(\cdot, s, t)\}_{s \in S}$ obeys single crossing differences (in (x, s)). When does the family $\{V(\cdot, s)\}_{s \in S}$ obey single crossing differences, where

$$V(x, s) = \int_T v(x, s, t)\lambda(t)\, dt$$

This is a different and harder problem from the first one we considered.

Optimization under uncertainty - Problem II

By definition, $V(x,s) = \int_T v(x,s,t)\lambda(t)\,dt$ obeys single crossing differences if for all $x'' > x'$,

$$\Delta(s) = \int_T [v(x'',s,t) - v(x',s,t)]\lambda(t)\,dt$$

is a single crossing function.

Define $\delta(s,t) = v(x'',s,t) - v(x',s,t)$, so

$$\Delta(s) = \int_T \delta(s,t)\lambda(t)dt$$

For each t, $\delta(\cdot,t)$ is a single crossing function (of s) because $\{v(\cdot,s,t)\}_{s\in S}$ obeys single crossing differences (in (x,s)).

So the issue is effectively this: when is the weighted sum of single crossing functions a single crossing function?

Summing single crossing functions

Let g and h, maps from the poset $(S, \geq)$ to $\mathbb{R}$, be single crossing functions.

Definition: The functions g and h are $\mathcal{S}$-summable (we write $f \sim g$), if $\alpha f + \beta g$ is a single crossing function for any positive scalars α and β.

Clearly $f \sim g$ if f and g are increasing functions. But there's more...

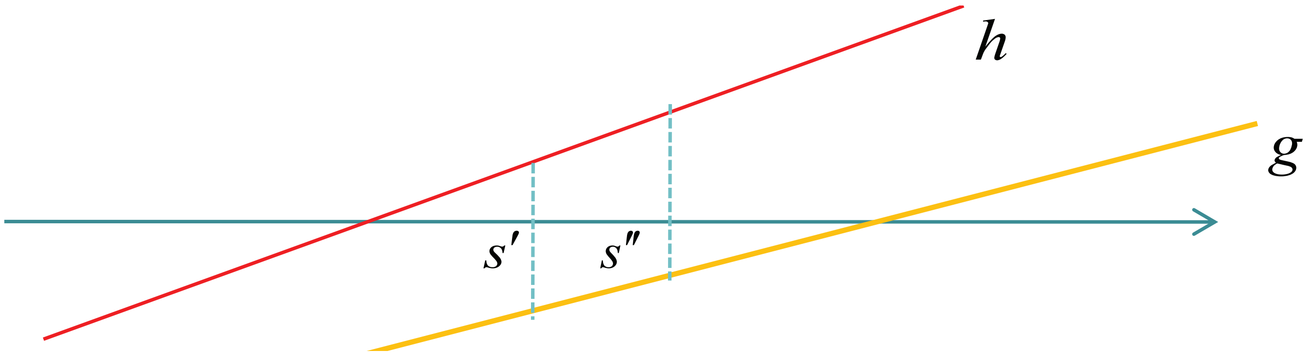
Proposition: The $\mathcal{S}$ functions h and g are $\mathcal{S}$-summable if and only if they have a monotone signed-ratio: whenever $h(s')$ and $g(s')$ are of opposite signs (with no loss of generality assume that $h(s') > 0$ and $g(s') < 0$), then for $s'' > s'$,

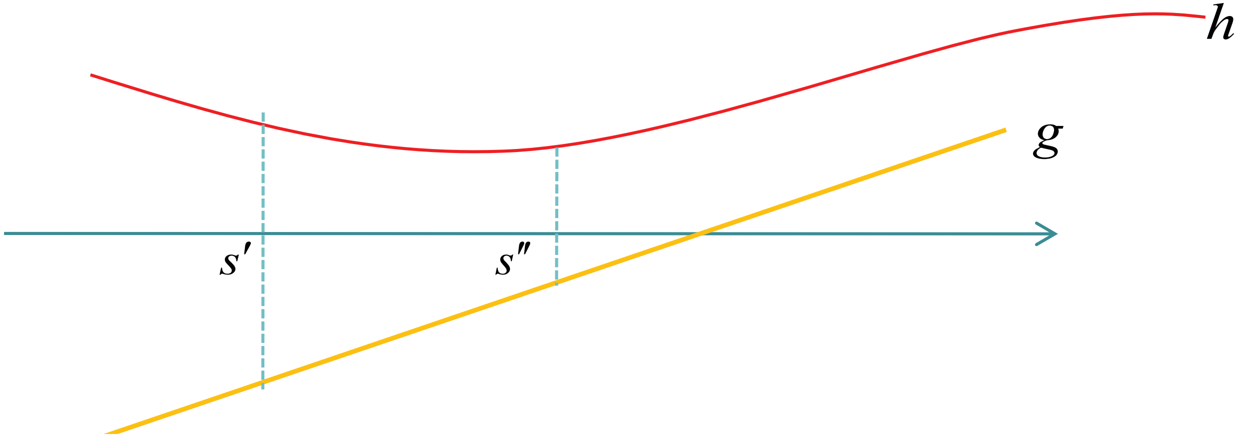
$$-\frac{g(s')}{h(s')} \geq -\frac{g(s'')}{h(s'')}$$

Summing single crossing functions

When $g(s') < 0$ and $h(s') > 0$, we require

$$-\frac{g(s')}{h(s')} \geq -\frac{g(s'')}{h(s'')}.$$





Summing single crossing functions

Example: Suppose $\mathbb{S} = \mathbb{R}$ and consider $h(s) = s^2 + 1$ and $g(s) = s^3$. In this case h is not an increasing function but we still have $h \sim g$. Clearly, h and g are of opposite signs only if $s < 0$; in that case,

$$-\frac{g(s)}{h(s)} = -\frac{s^3}{(s^2+1)}$$

is decreasing in s, for $s \in (-\infty, 0)$.

Lemma: Let f, g, and h be three $\mathcal{S}$ functions that are pairwise $\mathcal{S}$-summable. Then f and $g + h$ are $\mathcal{S}$-summable.

Let $T \subset \mathbb{R}^n$. We say that a family of $\mathcal{S}$ functions $\{g(\cdot;t)\}_{t \in T}$ (defined on $\mathbb{S} \subset \mathbb{R}^l$) form a related family if $g(\cdot;t) \sim g(\cdot;\tilde{t})$ for any t and $\tilde{t}$ in T.

Theorem 2: Let $\{g(\cdot;t)\}_{t \in T}$ be a related family of $\mathcal{S}$ functions. Then

$$G(s) = \int_T g(s;t)\lambda(t)\,dt \text{ is an } \mathcal{S} \text{ function.}$$

Optimization under uncertainty - Problem II

Corollary 2: Suppose that, for any fixed t, $v(x,s,t)$ has single crossing differences in (x,s). Then $V(x,s) = \int_T v(x,s,t)\lambda(t)\,dt$ has single crossing differences if, for any $x'' > x'$ and $\tilde{t}$ and $\hat{t}$,

$$\delta(\cdot,\tilde{t}) \sim \delta(\cdot,\hat{t}) \text{ where}$$

$\delta(s,\tilde{t}) = u(x'',s,\tilde{t}) - u(x',s,\tilde{t})$ and $\delta(s,\hat{t}) = u(x'',s,\hat{t}) - u(x',s,\hat{t})$.

To apply this corollary, we need to check, in the specific application, that $\delta(\cdot,\hat{t}) \sim \delta(\cdot,\tilde{t})$, typically by using the monotone signed-ratio condition, i.e., if $\delta(s^*,\tilde{t}) > 0$ and $\delta(s^*,\hat{t}) > 0$, then

$$\begin{aligned} -\frac{\delta(s^*,\hat{t})}{\delta(s^*,\tilde{t})} &= \frac{-u(x'',s^*,\hat{t}) + u(x',s^*,\hat{t})}{u(x'',s^*,\tilde{t}) - u(x',s^*,\tilde{t})} \\ &\geq \frac{-u(x'',s^{**},\hat{t}) + u(x',s^{**},\hat{t})}{u(x'',s^{**},\tilde{t}) - u(x',s^{**},\tilde{t})} \\ &= -\frac{\delta(s^{**},\hat{t})}{\delta(s^{**},\tilde{t})} \text{ for } s^{**} > s^*. \end{aligned}$$

Summing single crossing functions

Application: The firm faces uncertainty in its inverse demand function. Formally, it's problem is to choose x to maximize

$$V(x, -c) = \int_T u(xP(x,t) - cx)\lambda(t)\,dt,$$

where $T \subset \mathbb{R}$. When is it true that higher c leads to lower output?

Applying the corollary, $\{V(\cdot, -c)\}_{c>0}$ has single crossing differences if, for any $\tilde{t}$ and $\hat{t}$, we have $\delta(\cdot, \tilde{t}) \sim \delta(\cdot, \hat{t})$, where

$$\delta(s,t) = u(x''P(x'',t) - cx'') - u(x'P(x',t) - cx').$$

This is true if

(i) u obeys DARA

(ii) P is decreasing in x, increasing in t, and logsupermodular in (x,t).

Monotone Bayesian games

In a Bayesian game, a player's strategy is a function that maps the signal he receives to an action.

Definition: A player's strategy in a Bayesian game is monotone if his action is always pure and it is increasing in the signal he receives.

Theorem: (Athey) A Bayesian game has an equilibrium in monotone strategies if the following holds:

whenever all other players are playing monotone strategies, a player has a best response that is also a monotone strategy.

Given Athey's theorem, the problem of equilibrium existence in Bayesian games reduces to establishing a comparative statics result.

Monotone Bayesian games

Consider an n-player Bayesian game where player i observes a signal $s_i \in \mathbb{S}_i \subset \mathbb{R}$. The joint density of $s = (s_1, s_2, ..., s_n)$ is given by the function λ. Player 1's posterior distribution after observing s_1 is denoted by $\lambda(s_2, s_3, ..., s_n | s_1)$.

The utility of Player 1 in state s and given actions $(x_1, x_2, ..., x_n)$ is denoted by $u((x_1, x_2, ..., x_n); s)$.

Suppose that for $i \geq 2$, player i has strategy ϕ_i, i.e., player i takes action $\phi_i(s_i) \in X_i$ after observing s_i.

Therefore, Player 1's expected utility if he takes action $x \in X_1$ after observing s_1 is

$$\int_{\mathbb{S}_n} ... \int_{\mathbb{S}_3} \int_{\mathbb{S}_2} u\left((x, \phi_2(s_2), \phi_3(s_3), ..., \phi_n(s_n))\,; s\right) \lambda(s_2, ..., s_n | s_1)\, ds_2 ... ds_n.$$

Suppose that ϕ_i is increasing for all $i \geq 2$; under what conditions will Player 1 also have an increasing strategy?

Optimization under uncertainty - Problem III

After observing s_1, Player 1 chooses x to maximize

$$V(x;s_1) = \int_{\mathbb{S}_{-1}} v(x;s_1,s_2,...,s_n)\lambda(s_2,s_3,...,s_n|s_1)ds_{-1}$$

where $v(x;s) = u\left((x,\phi_2(s_2),\phi_3(s_3),...,\phi_n(s_n));s_1,s_2,...,s_n\right).$

We know that $\text{argmax}_{x\in X}V(x;s_1)$ increases with s_1 if, for $x'' > x'$, $\Delta : \mathbb{S}_1 \to \mathbb{R}$ is a single crossing function, where

$$\begin{aligned}
\Delta(s_1) &= V(x'';s_1) - V(x';s_1) \\
&= \int_{\mathbb{S}_{-1}} [v(x'';s) - v(x';s)]\lambda(s_2,s_3,...,s_n|s_1)\,ds_{-1} \\
&= \int_{\mathbb{S}_{-1}} \delta(s_1,s_2,...,s_n)\lambda(s_2,s_3,...,s_n|s_1)\,ds_{-1}.
\end{aligned}$$

This problem combines problems I and II and more...

Optimization under uncertainty - Problem III

We assume that the signals are affiliated, i.e., λ is a logsupermodular function.

Conveniently, this also implies that $\lambda(s_2, s_3, ..., s_n|s_1)$ is a logsupermodular function of $(s_1, s_2, ...s_n)$.

Definition: Let $\mathbb{S}_i \subset \mathbb{R}$ for $i = 1, 2, ..., n$ and $\mathbb{S} = \Pi_{i=1}^{n} \mathbb{S}_i$. A single crossing function $\delta : \mathbb{S} \to \mathbb{R}$ is an 1-integrable single crossing function if

$$\delta(\cdot, s'_K) \sim \delta(\cdot, s''_K) \text{ where } s'_K < s''_K \text{ and}$$

K' and K'' are subsets of $\{1, 2, ..., n\} \setminus \{1\}$.

Theorem 3: Suppose $\delta : \mathbb{S} \to \mathbb{R}$ is an 1-integrable single crossing function and λ is a logsupermodular function. Then

$$\Delta(s_1) = \int_{\mathbb{S}_{-1}} \delta(s_1, s_2, ..., s_n)\lambda(s_2, s_3, ..., s_n|s_1)\, ds_{-1}$$

is a single crossing function.

Optimization under uncertainty - Problem III

Theorem 3: Suppose $\delta : \mathbb{S} \to \mathbb{R}$ is an 1-integrable single crossing function and λ is a logsupermodular function. Then

$$\Delta(s_1) = \int_{\mathbb{S}_{-1}} \delta(s_1, s_2, ..., s_n)\lambda(s_2, s_3, ..., s_n|s_1)\, ds_{-1}$$

is a single crossing function.

Recall Theorem 1 says that $\Delta(s_1) = \int_{S_2} \delta(s_2)\lambda(s_2|s_1)\, ds_2$ is a single crossing function if λ is logsupermodular and δ is a single crossing function. So Theorem 3 generalizes Theorem 1 to multiple dimensions.

Corollary 3: Suppose λ is logsupermodular and $\{v(\cdot, s)\}_{s\in S}$ has the following property: for any $x'' > x'$, $\delta(s) = u(x'', s) - u(x', s)$ is a 1-integrable single crossing function. Then $\{V(\cdot, s_1)\}_{s_1\in S_1}$ obeys single crossing differences, where

$$V(x, s_1) = \int_{\mathbb{S}_{-1}} v(x_1; s_1, s_2, ..., s_n)\lambda(s_2, s_3, ..., s_n|s_1) ds_{-1}.$$

Application: equilibrium in first-price auctions

Let $u_i(b,s)$ denote the payoff to bidder i if he obtains the object after paying bid b and the signal is $s = (s_1, s_2, ..., s_n) \in \mathbb{S} = \Pi_{i=1}^n \mathbb{S}_i$.

Theorem 4: A first-price auction has an equilibrium in monotone bidding strategies if the following assumptions hold:

(1) the joint density function λ over signals $s = (s_1, s_2, ..., s_n)$ is logsupermodular;

for bidder i ($i = 1, 2, ..., n$),

(2) $u_i(b; s)$ is increasing in s and decreasing in b

(3) for any $b'' > b'$, $K \subset \{1, 3, ..., n\} \setminus \{i\}$, $s''_K > s'_K$,

$$u_i(b''; \cdot, s'_K) - u_i(b'; \cdot, s'_K) \sim u_i(b''; \cdot, s''_K).$$

Equilibrium in first-price auctions

(3) for any $b'' > b'$, $K \subset \{1, 3, ..., n\} \setminus \{i\}$, $s''_K > s'_K$,

$$u_i(b''; \cdot, s'_K) - u_i(b'; \cdot, s'_K) \sim u_i(b''; \cdot, s''_K).$$

Example: Suppose that u_i has increasing differences in (b, s). Then

$$u_i(b''; \cdot, s'_K) - u_i(b'; \cdot, s'_K) \text{ is an increasing function of } s_{N \setminus K}.$$

Since u_i is increasing in s, $u_i(b''; \cdot, s''_K)$ is an increasing function of $s_{N \setminus K}$. So clearly, (3) holds.

Example: Assume private values, so $u_i(b, s) = u_i(b, s_i)$. Then

$$L = \{(b, s) : u_i(b, s) > 0\} \text{ is a sublattice of } R^2.$$

Suppose that u_i is a losupermodular function in L. Then we can check that (3) holds by checking that the functions have a monotone signed-ratio:

$$\frac{u_i(b', s_i) - u_i(b'', s_i)}{u_i(b'', s_i)} = \frac{u_i(b', s_i)}{u_i(b'', s_i)} - 1 \text{ is decreasing in } s_i.$$

Equilibrium in first-price auctions

(3) for any $b'' > b'$, $K \subset \{1, 3, ..., n\} \setminus \{i\}$, $s''_K > s'_K$,

$$u_i(b''; \cdot, s'_K) - u_i(b'; \cdot, s'_K) \sim u_i(b''; \cdot, s''_K).$$

Example: Suppose $u_i(b; s) = (Y - b)\phi(s) - Y$, where ϕ is increasing.

Note that $u_i(b''; s) - u_i(b'; s) = (b' - b'')\phi(s)$, which is *decreasing* in s if ϕ is increasing. So u_i does not obey increasing differences.

However, it is straightforward to check that

(3) for any $b'' > b'$, $K \subset \{1, 3, ..., n\} \setminus \{i\}$, $s''_K > s'_K$,

$$u_i(b''; \cdot, s'_K) - u_i(b'; \cdot, s'_K) \sim u_i(b''; \cdot, s''_K).$$

holds id ϕ is increasing and logsupermodular.

Equilibrium in first-price auctions

Theorem 4: A first-price auction has an equilibrium in monotone bidding strategies if the following assumptions hold:

(1) the joint density function λ over signals $s = (s_1, s_2, ..., s_n)$ is logsupermodular;
for bidder i ($i = 1, 2, ..., n$),
(2) $u_i(b; s)$ is increasing in s and decreasing in b
(3) for any $b'' > b'$, $K \subset \{1, 3, ..., n\} \setminus \{i\}$, $s''_K > s'_K$,

$$u_i(b''; \cdot, s'_K) - u_i(b'; \cdot, s'_K) \sim u_i(b''; \cdot, s''_K).$$

'proof': The proof relies on the existence theorem of Athey for Bayesian games with monotone strategies. The main thing we need to do is to show the following:

when other players play bidding strategies that are increasing in the signals they receive, then bidder 1 will also choose a bidding strategy that increases with his signal.

Monotone bidding strategies in first-price auctions

Suppose there are two bidders and bidder 2's strategy is ϕ_2.

With bid b'' by Player 1, and ignoring the possibility of ties, Player 1 wins the object if and only if $\phi_2(s_2) < b''$.
Let $s_2^{**} = \sup\{s_2 \in \mathbb{S}_2 : \phi_2(s_2) < b''\}$. Then

$$U_1(b''; s_1) = \int_0^{s_2^{**}} u_1(b''; s)\lambda(s_2|s_1)\, ds_2.$$

Similarly, with bid $b' < b''$, we have $U_1(b'; s_1) = \int_0^{s_2^*} u_1(b'; s)\lambda(s_2|s_1)\, ds_2$,
where $s_2^* = \sup\{s \in \mathbb{S}_2 : \phi_2(s_2) < b'\}$.
Note that $s_2^* \leq s_2^{**}$.

Player 1's optimal bid increases with s_1 so long as

$\Delta(s_1) = U_1(b''; s_1) - U_1(b'; s_1)$ is a single crossing function.

Monotone bidding strategies in first-price auctions

$$\text{So } \Delta(s_1) = U_1(b''; s_1) - U_1(b'; s_1)$$
$$= \int_0^{s_2^{**}} u_1(b''; s)\lambda(s_2|s_1)\, ds_2 - \int_0^{s_2^*} u_1(b'; s)\lambda(s_2|s_1)\, ds_2$$
$$= \int_0^{s_2^{**}} \delta(s_1, s_2)\lambda(s_2|s_1)\, ds_2 \text{ where}$$

$\delta(s_1, s_2) = u_1(b''; s_1, s_2)$ for $s_2 \in [s_2^*, s_2^{**}]$ and
$\delta(s_1, s_2) = u_1(b''; s_1, s_2) - u_1(b'; s_1, s_2)$ for $s_2 \in [0, s_2^*]$.

By Theorem 1, Δ is a single crossing function if

$$\delta(\cdot, \hat{s}_2) \sim \delta(\cdot, \tilde{s}_2) \text{ for any } \hat{s}_2 \text{ and } \tilde{s}_2 \text{ in } [0, s_2^{**}].$$

Monotone bidding strategies in first-price auctions

So $\Delta(s_1) = U_1(b''; s_1) - U_1(b'; s_1) = \int_0^{s_2^{**}} \delta(s_1, s_2)\lambda(s_2|s_1)\, ds_2$ where

$\delta(s_1, s_2) = u_1(b''; s_1, s_2)$ for $s_2 \in [s_2^*, s_2^{**}]$ and
$\delta(s_1, s_2) = u_1(b''; s_1, s_2) - u_1(b'; s_1, s_2)$ for $s_2 \in [0, s_2^*]$.

We need to check that ($\star$) $\delta(\cdot, \hat{s}_2) \sim \delta(\cdot, \tilde{s}_2)$ for any $\hat{s}_2$ and $\tilde{s}_2$.

If $\hat{s}_2$, $\tilde{s}_2 > s_2^*$, ($\star$) holds if $u_1(b''; \cdot, \hat{s}_2) \sim u_1(b''; \cdot, \tilde{s}_2)$, which is true since u_1 is increasing in s_1.

If $\hat{s}_2$, $\tilde{s}_2 < s_2^*$, ($\star$) holds if

$$u_1(b''; \cdot, \hat{s}_2) - u_1(b'; \cdot, \hat{s}_2) \sim u_1(b''; \cdot, \tilde{s}_2) - u_1(b'; \cdot, \tilde{s}_2),$$

which is true since both sides are negative.

If $\hat{s}_2 < s_2^* < \tilde{s}_2$, ($\star$) holds if

$$u_1(b''; \cdot, \hat{s}_2) - u_1(b'; \cdot, \hat{s}_2) \sim u_1(b''; \cdot, \tilde{s}_2)$$

which is condition (4′). **QED**

Comparing SCD and IDO

Before we discuss comparative information, I will return to the concept of interval dominance order we first considered last week.

Definition: The family $\{v(\cdot; s)\}_{s\in S}$ obeys the single crossing differences if for any $x^{**} > x^*$ and $s'' > s'$

$$v(x^{**}; s') - v(x^*; s') \geq (>)\, 0 \implies v(x^{**}; s'') - v(x^*; s'') \geq (>)\, 0.$$

Equivalently, the family $\{v(\cdot; s)\}_{s\in S}$ obeys SCD if on any pair of actions $\{x^*, x^{**}\}$ (with $x^{**} > x^*$) and $s'' > s'$,

(i) $x^{**} \in$ argmax$_{x\in\{x^*,x^{**}\}} v(x, s') \implies x^{**} \in$ argmax$_{x\in\{x^*,x^{**}\}} v(x, s'')$.

(ii) if $x^{**} \in$ argmax$_{x\in\{x^*,x^{**}\}} v(x, s')$ and $x^* \notin$ argmax$_{x\in\{x^*,x^{**}\}} v(x, s')$ $\implies x^{**} \in$ argmax$_{x\in\{x^*,x^{**}\}} v(x, s'')$ and $x^* \notin$ argmax$_{x\in\{x^*,x^{**}\}} v(x, s'')$.

Comparing SCD and IDO

Definition: The family $\{v(\cdot; s)\}_{s\in S}$ obeys the interval dominance order if for any $x'' > x'$ and $s'' > s'$, such that

$v(x''; s') - v(x; s') \geq 0$ for all $x \in [x', x'']$, we have

$$f(x''; s') - f(x'; s') \geq (>)\, 0 \implies f(x''; s'') - f(x'; s'') \geq (>)\, 0.$$

Equivalently, the family $\{v(\cdot; s)\}_{s\in S}$ obeys IDO if on any interval $[x^*, x^{**}]$ and $s'' > s'$,

(i) $x^{**} \in \text{argmax}_{x\in[x^*,x^{**}]} v(x, s') \implies x^{**} \in \text{argmax}_{x\in[x^*,x^{**}]} v(x, s'')$.

(ii) if $x^{**} \in \text{argmax}_{x\in[x^*,x^{**}]} v(x, s')$ and $x^* \notin \text{argmax}_{x\in[x^*,x^{**}]} v(x, s')$
$\implies x^{**} \in \text{argmax}_{x\in[x^*,x^{**}]} v(x, s'')$ and $x^* \notin \text{argmax}_{x\in[x^*,x^{**}]} v(x, s'')$.

Compared to the definition of SCD, we have replaced the pair $\{x^*, x^{**}\}$ with the interval $[x^*, x^{**}]$.

Problem I revisited

We return to the case where $S \subset \mathbb{R}$.

Recall Corollary 1: Suppose that $\{v(\cdot;s)\}_{s\in S}$ obeys single crossing differences and $\{\lambda(\cdot,\theta)\}_{\theta\in\Theta}$ obeys the monotone likelihood ratio order. Then $\{V(\cdot;s)\}_{\theta\in\Theta}$ obeys single crossing differences, where

$$V(x,\theta) = \int_S v(x,s)\lambda(s,\theta)\,ds.$$

Consequently, $\text{argmax}_{x\in X} V(x;\theta)$ is increasing in θ.

Theorem 5: Suppose that $\{v(\cdot;s)\}_{s\in S}$ obeys the interval dominance order and $\{\lambda(\cdot,\theta)\}_{\theta\in\Theta}$ obeys the monotone likelihood ratio order. Then $\{V(\cdot;s)\}_{\theta\in\Theta}$ obeys the interval dominance order, where

$$V(x,\theta) = \int_S v(x,s)\lambda(s,\theta)\,ds.$$

Consequently, $\text{argmax}_{x\in X} V(x;\theta)$ is increasing in θ.

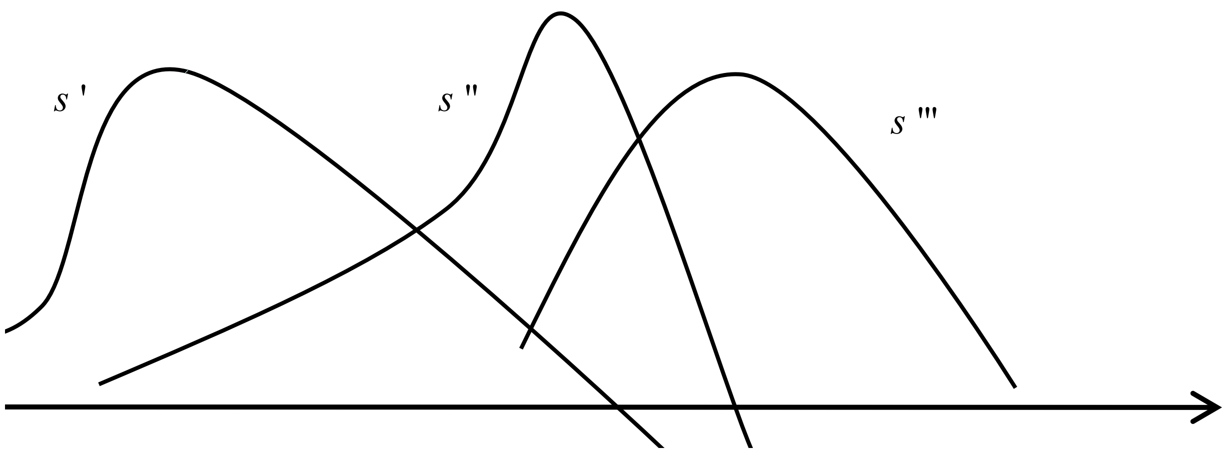
Problem I and IDO

Theorem 5: Suppose that $\{v(\cdot;s)\}_{s\in S}$ obeys the interval dominance order and $\{\lambda(\cdot,\theta)\}_{\theta\in\Theta}$ obeys the monotone likelihood ratio order. Then $\{V(\cdot;s)\}_{\theta\in\Theta}$ obeys the interval dominance order, where

$$V(x,\theta) = \int_S v(x,s)\lambda(s,\theta)\,ds.$$

Consequently, $\text{argmax}_{x\in X} V(x;\theta)$ is increasing in θ.

A family of quasiconcave functions parameterized by their peaks obeys the interval dominance order, but not necessarily single crossing differences.



Problem I and IDO

Theorem 5: Suppose that $\{v(\cdot;s)\}_{s\in S}$ obeys the interval dominance order and $\{\lambda(\cdot,\theta)\}_{\theta\in\Theta}$ obeys the monotone likelihood ratio order. Then $\{V(\cdot;s)\}_{\theta\in\Theta}$ obeys the interval dominance order, where

$$V(x,\theta) = \int_S v(x,s)\lambda(s,\theta)\,ds.$$

Consequently, $\text{argmax}_{x\in X} V(x;\theta)$ is increasing in θ.

Corollary 3: Suppose that $\{v(\cdot;s)\}_{s\in S}$ is a family of quasiconcave functions parameterized by their peaks and $\{\lambda(\cdot,\theta)\}_{\theta\in\Theta}$ obeys the monotone likelihood ratio order.

Then $\{V(\cdot;s)\}_{\theta\in\Theta}$ obeys the interval dominance order. Consequently, $\text{argmax}_{x\in X} V(x;\theta)$ is increasing in θ.

Note the difficulty in establishing this result: the sum of quasiconcave functions is not quasiconcave.

Information structures

Setting: Agent chooses action after observing a signal $z \in Z$, an interval of $\mathbb{R}$, but before realization of state.

Distribution over signals at a state s is $H(z|s)$ (with density $h(z|s)$).

The family $\{h(\cdot|s)\}_{s \in S}$ is the information structure H.

Assume that distributions are MLR-ordered, i.e., for $s'' > s'$

$$\frac{h(z|s'')}{h(z|s')} \text{ is increasing in } z.$$

Higher states make higher signals more likely.

Suppose the agent is Bayesian, i.e., he has a unique prior P on the S.

Given P, the agent can work out the posterior distributions $\{\tilde{h}_P(\cdot|z)\}_{z \in Z}$.

Comparing Information Structures

Fact: If $\{h(\cdot|s)\}_{s\in S}$ is MLR-ordered family then $\{\tilde{h}_P(\cdot|z)\}_{z\in Z}$ is also an MLR-ordered family.
Higher signals make higher states more likely.

Corollary 4: Suppose $\{h(\cdot|s)\}_{s\in S}$ is MLR-ordered and $\{v(\cdot,s)\}_{s\in S}$ obeys the interval dominance order. Then, for any prior P, agent has an increasing decision rule, i.e., there is

$$\phi(z) \in \text{argmax}_{x\in X} \int_{s\in S} v(x,s)\tilde{h}_P(s,z)ds$$

such that ϕ is increasing in z.
In other words, higher signals lead to higher actions.

Proof: Since $\{\tilde{h}_P(\cdot|z)\}_{z\in Z}$ is an MLR-ordered family, result follows immediately from Theorem 5. **QED**

Comparing Information Structures

Assume prior is P and consider information structure H. If ϕ_H is the optimal decision rule, then agent's ex ante utility is

$$\mathcal{V}(H,P) = \int_{z\in Z} \left[\int_{s\in S} v(\phi_H(z), s) d\tilde{h}_P(s|z) \right] d\nu_H$$

where ν_H is the marginal distribution of z.

We wish to compare H with another information structure $G = \{g(\cdot|s)\}_{s\in S}$. Suppose its optimal decision rule is ϕ_G, so

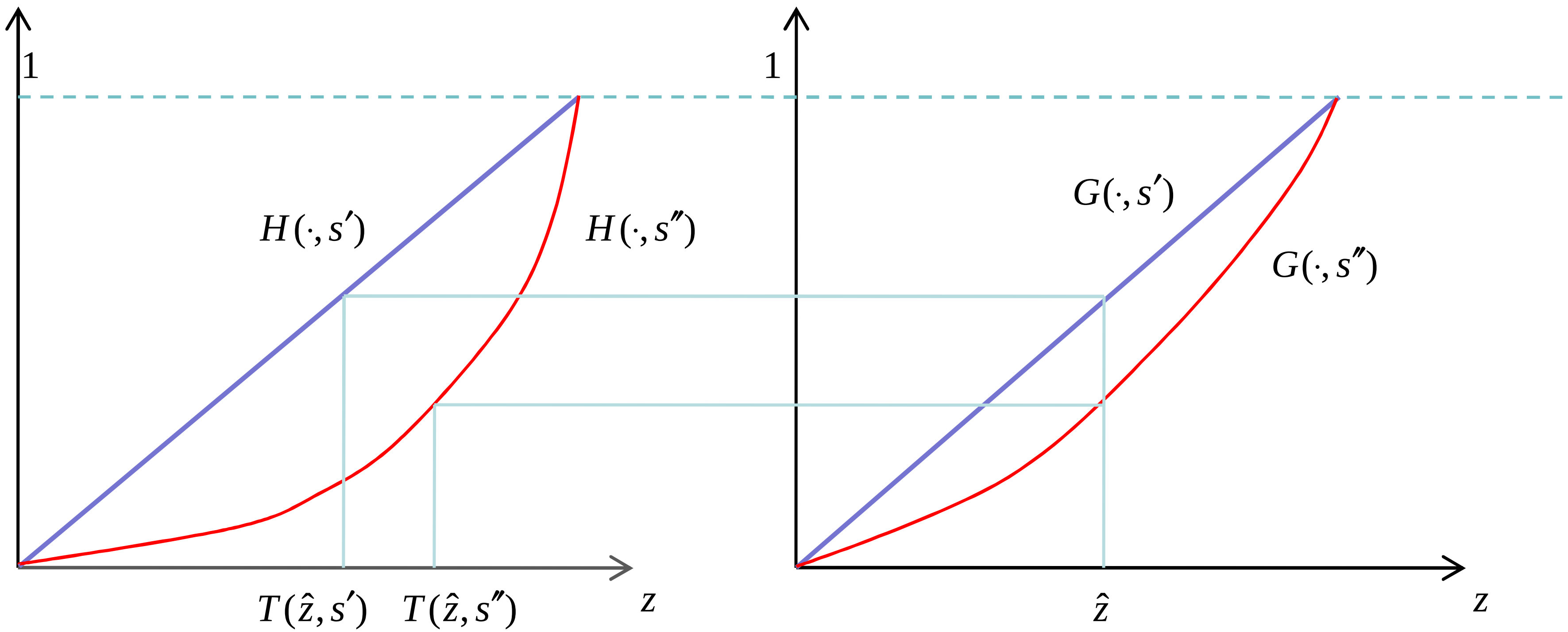
$$\mathcal{V}(G,P) = \int_{z\in Z} \left[\int_{s\in S} v(\phi_G(z), s) d\tilde{g}_P(s|z) \right] d\nu_G.$$

When is $\mathcal{V}(H,P) \geq \mathcal{V}(G,P)$ for all P?

Lehmann informativeness

Recall, $H(\cdot|s)$ is the distribution of signal z conditional on state s.
Similarly, $G(\cdot|s)$ is the distribution of signal z conditional on state s.
Define $T(z,s)$ by $H(T(z,s)|s) = G(z|s)$.

Definition: H is more informative than G (in the sense of Lehmann) if $T(z,\cdot)$ is increasing in s.



Comparing Information Structures

Theorem 6: Suppose that

(i) information structure H is Lehmann-more informative than G
(ii) $\{v(\cdot, s)\}_{s \in S}$ obeys the interval dominance order
(iii) at the prior P, the optimal decision rule for G, ϕ_G, is increasing in z
(assuming (ii), this condition holds if $H = \{h(\cdot|s)\}_{s \in S}$ is MLR-ordered.)

Then $\mathcal{V}(H, P) \geq \mathcal{V}(G, P)$.

This result extends Lehmann, who reached the same conclusion for a quasi-concave family parameterized by their peaks.

References

Listed below are readings containing the results discussed in the lecture.

Aggregating the single crossing property, Problems I, II, and III. (Theorems 1, 2, and 3)
Quah, J. K.-H. and B. Strulovici (2010): “Aggregating the single crossing property: theory and applications to comparative statics and Bayesian games," *Working Paper, Department of Economics, Oxford,* No. 493.

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Comparing information structures (Theorem 6)
Lehmann, E. L. (1988): “Comparing Location Experiments," *The Annals of Statistics,* 16(2), 521-533.
Quah, J. K.-H. and B. Strulovici (2009) (as cited above)