Survey Lecture on Revealed Preference Analysis

Part I

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Afriat's Theorem

Let $\mathcal{O} = \{(p^t, x^t)\}_{t \in T}$ be a set of observations drawn from a consumer.

Each observation consists of price vector $p^t = (p_1^t, p_2^t, ..., p_\ell^t) \gg 0$ and consumption bundle $x^t = (x_1^t, x_2^t, ..., x_\ell^t) \geqslant 0$ chosen by consumer at p^t.

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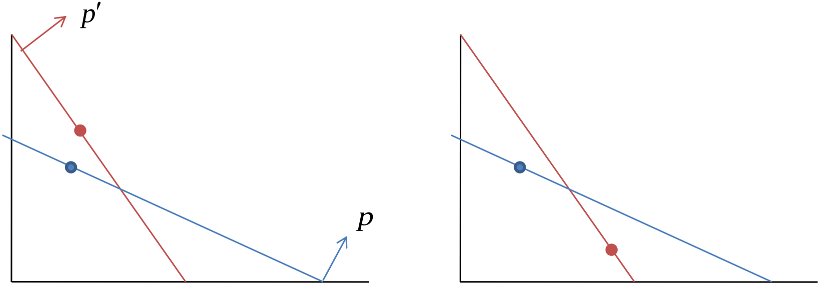
Afriat's Theorem provides an answer to the following question:

what conditions on $\mathcal{O}$ are necessary and sufficient for it to arise from a consumer maximizing a 'well-behaved' utility function?

It is clear that some data sets are consistent with utility maximization while others are not.



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In other words, utility-maximization has *observable restrictions.*
What restrictions on the data are *necessary and sufficient* for us to recover a utility function generating the data?

Afriat's Theorem: *A data set* $\mathcal{O} = \{(p^t, x^t)\}_{t \in T}$ *is consistent with utility maximization if and only if it obeys the generalized axiom of revealed preference (GARP).*

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Definition: A function $U : \mathbb{R}^\ell_+ \to \mathbb{R}$ is said to rationalize the set $\mathcal{O}$ if, at every observation $t \in T$,

$$U(x^t) \geqslant U(x) \text{ for all } x \in \{x \in \mathbb{R}^\ell_+ : p^t \cdot x \leqslant p^t \cdot x^t\}.$$

A data set $\mathcal{O}$ that admits rationalization by a locally non-satiated utility function U is said to be consistent with utility-maximization or rationalizable.

What is GARP?

Denote the set of observed demands by $\mathcal{X}$, i.e., $\mathcal{X} = \{x^t\}_{t \in T}$.

For x^t, $x^s \in \mathcal{X}$, x^t is directly revealed preferred to x^s if $p^t \cdot x^s \leqslant p^t \cdot x^t$. [Notation: $x^t \succeq^* x^s$.]

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$$x^{t_1} \succeq^* x^{t_2}, \; x^{t_2} \succeq^* x^{t_3}, \; \ldots, x^{t_{n-1}} \succeq^* x^{t_n}, \text{ and } x^{t_n} \succeq^* x^{t_1},$$

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Alternatively, if

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GARP says that the only RP cycles are *weak* RP cycles.

The necessity of GARP

Proposition 1: *Suppose that $\mathcal{O}$ is drawn from a consumer with a locally non-satiated utility function. Then it must satisfy GARP.*

Proof: By definition, if $x^t \succeq^* x^s$, then $p^t \cdot x^s \leqslant p^t \cdot x^t$ so utility maximization implies that $U(x^s) \leqslant U(x^t)$.

If $x^t \succ\!\!\succ^* x^s$, then $p^t \cdot x^s < p^t \cdot x^t$ so maximization of a locally non-satiated utility function implies that $U(x^s) < U(x^t)$.

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Therefore, if

$$x^{t_1} \succeq^* x^{t_2},\ x^{t_2} \succeq^* x^{t_3},\ \ldots,\ x^{t_{n-1}} \succeq^* x^{t_n}, \text{ and } x^{t_n} \succeq^* x^{t_1},$$

then

$$U(x^{t_1}) \geqslant U(x^{t_2}) \geqslant U(x^{t_3}) \geqslant \ldots \geqslant U(x^{t_{n-1}}) \geqslant U(x^{t_n}) \geqslant U(x^{t_1})$$

and none of the inequalities can be strict. Therefore,

$$x^{t_1} \not\succ^* x^{t_2},\ x^{t_2} \not\succ^* x^{t_3},\ \ldots,\ x^{t_{n-1}} \not\succ^* x^{t_n}, \text{ and } x^{t_n} \not\succ^* x^{t_1}$$

as required by GARP. *QED*

The role of GARP

It is quite obvious that GARP is necessary and sufficient for the existence of a preference $\succeq$, i.e., a transitive, reflexive and complete relation, on $\mathcal{X} = \{x^t\}_{t\in T}$ that is consistent with the revealed relations $\succeq^*$ and $\gg^*$, i.e.,

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Afriat's Theorem: (restated) *Suppose $\mathcal{O} = \{(p^t, x^t)\}_{t \in T}$ admits a preference $\succeq$ on $\mathcal{X} = \{x^t\}_{t \in T}$ that is consistent with $\succeq^*$ and $\succ\!\!\succ^*$.*

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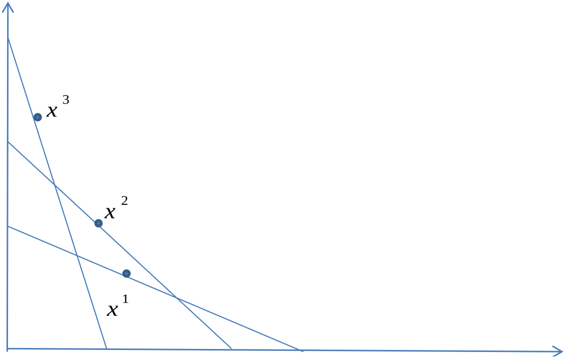
Then there exists a strictly increasing, continuous, and concave utility function $U : \mathbb{R}^\ell_+ \to \mathbb{R}$ that rationalizes the data and extends $\succeq$, *in the following sense:*

$$U(x^t) \geqslant (>)\, U(x^s) \textit{ if and only if } x^t \succeq (\succ)\, x^s.$$

Afriat's Theorem

Example: $\mathcal{O} = \{(p^t, x^t)\}_{t=1}^3$, $\mathcal{X} = \{x^t\}_{t=1}^3$, and $x^2 \ggcurly^* x^1$.



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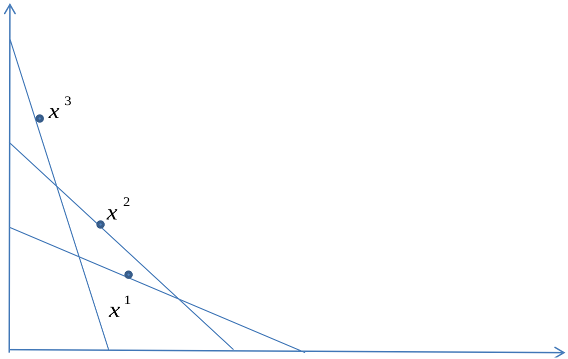

There are five preferences on $\mathcal{X} = \{x^1, x^2, x^3\}$ that extend $x^2 \ggcurly^* x^1$.

For each preference $\succeq$, there is $U : \mathbb{R}^2_+ \to \mathbb{R}$ that agrees with $\succeq$ and rationalizes the data.

The sufficiency of GARP

Proposition 2: Suppose $\mathcal{O} = \{(p^t, x^t)\}_{t \in T}$ obeys GARP.

1. Then there are numbers ϕ^t and $\lambda^t > 0$ (for every $t \in T$) that solve the linear inequalities:

$$\phi^t \leqslant \phi^k + \lambda^k p^k \cdot (x^t - x^k) \text{ for all } k \neq t.$$

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2. The data $\mathcal{O}$ can be rationalized by the utility function

$$U(x) = \min_{(p^t, x^t) \in \mathcal{O}} \left\{ \phi^t + \lambda^t \, p^t \cdot (x - x^t) \right\}.$$

This utility function is continuous, strictly increasing, concave, and satisfies $U(x^t) = \phi^t$.

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Notice that the implementation of Afriat's Theorem is computationally very straightforward. We could check GARP directly (using Warshall's algorithm).

Or we could simply solve the linear inequalities

$$\phi^t \leqslant \phi^k + \lambda^k p^k \cdot (x^t - x^k) \text{ for all } k \neq t.$$

This at least partly accounts for its use in empirical work.

Afriat's Theorem can also be generalized to non-linear constraint sets. Linear prices is not at all important to the result.

Afriat's Theorem generalized, part I

Suppose that each observation consists of a chosen bundle x^t in the constraint set K^t, where $x^t \in \mathbb{R}^\ell_+$ and K^t is a compact subset of $\mathbb{R}^\ell_+$ that is downward inclusive in the following sense:

if $y \in \mathbb{R}^\ell_+$ and $y \leqslant x$ for some $x \in K^t$, then $y \in K^t$.

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then we cannot replace $\geq^*$ with $\gg^*$ anywhere in this sequence.

It is clear that if the agent is maximizing a strictly increasing function utility function, then GARP holds.

The converse is also true but less obvious.

Theorem: (Forges-Minelli) *A data set $\mathcal{O} = \{(K^t, x^t)\}_{t \in T}$ is consistent with the maximization of a strictly increasing and continuous utility function if and only if it obeys GARP.*

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Notice that, unlike the case with linear budget sets, the theorem no longer guarantees that the utility function is concave. And it fact it *cannot* be guaranteed.

Implementation

- Data from the portfolio choice experiment in Choi, Fisman, Gale, and Kariv (AER, 2007).
- 93 undergraduate subjects participated in the experiment at UC Berkeley, each completing 50 decision problems under risk.
- There were two states of the world, each occurring with a known probability, and two Arrow-Debreu securities, one for each state.
- In each decision problem, every subject was given a budget; income was normalized to one and state prices were chosen at random.
- 47 subjects were subjected to the symmetric treatment, where $\pi_1 = \pi_2 = 1/2$. The rest had the asymmetric treatment where $\pi_1 = 1/3$, $\pi_2 = 2/3$ (or vice versa).

Implementation

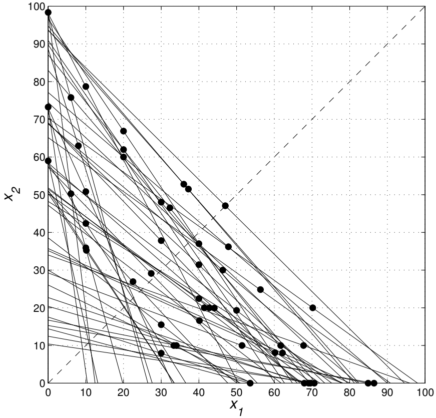

Figure: Subject 216 (symmetric treatment)

In the symmetric treatment, 12/47 pass GARP.
In the asymmetric treatment, 4/46 pas GARP.

Roughly, 17% pass GARP.

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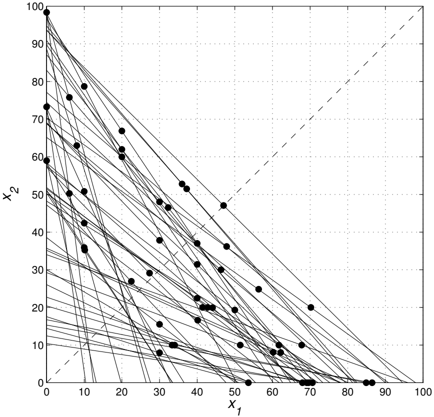

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Roughly, 17% pass GARP. What are we to make of everyone else?

Critical Cost Efficiency Index

We use the approach suggested Afriat (1972) and Varian (1990).

A data set $\mathcal{O} = \{(p^t, x^t)\}_{t \in \mathcal{T}}$ is rationalizable by some family **U** if there is a function $U : \mathbb{R}^\ell_+ \to \mathbb{R}$ belonging to **U** such that

$$U(x^t) \geqslant U(x) \text{ for all } x \in \mathcal{B}^t = \{x \in \mathbb{R}^\ell_+ : p^t \cdot x \leqslant p^t \cdot x^t\}.$$

Suppose that no function belonging **U** rationalizes the data. We could then make the requirement less stringent by shrinking all the budget sets in $\mathcal{O}$ by an efficiency index α in $[0, 1)$:

we find U in **U** such that $U(x^t) \geqslant U(x)$ for all $x \in \mathcal{B}^t(\alpha)$, where

$$\mathcal{B}^t(\alpha) = \{x \in \mathbb{R}^\ell_+ : x \leqslant x^t\} \cup \{x \in \mathbb{R}^\ell_+ : p^t \cdot x \leqslant \alpha\, p^t \cdot x^t\}.$$

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The *largest* α at which the data passes the test is known as the critical cost efficiency index (CCEI) associated with $\mathcal{O}$ and **U**.

Critical Cost Efficiency Index

A bit more formally: given a family of utility functions $\mathbf{U}$, the CCEI of a data set $\mathcal{O} = \{(p^t, x^t)\}_{t \in T}$ is

$$\alpha^* = \sup\{\alpha : \alpha \in A\},$$

where $\alpha \in A$ if there is a function $U : \mathbb{R}^\ell_+ \to \mathbb{R}$ belonging to $\mathbf{U}$ such that $U(x^t) \geqslant U(x)$ for all $x \in \mathcal{B}^t(\alpha)$, where

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In other words, there is $U \in \mathbf{U}$ that rationalizes the data set $\mathcal{O}(\alpha^*) = \{(\mathcal{B}^t(\alpha^*), x^t)\}_{t \in T}$.

Now suppose $\mathbf{U}$ is the family of all strictly increasing utility and continuous utility functions. Whether there is $U \in \mathbf{U}$ that rationalizes the data set $\mathcal{O}(\alpha^*)$ is then equivalent to whether $\mathcal{O}(\alpha^*)$ obeys GARP (by the Forges-Minelli Theorem).

GARP can be directly checked using Warshall's algorithm.

Notice that $\mathcal{B}^t(\alpha)$ is not a convex set.

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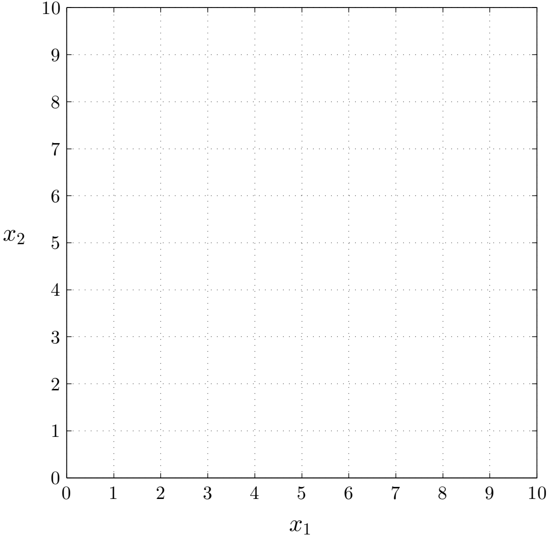
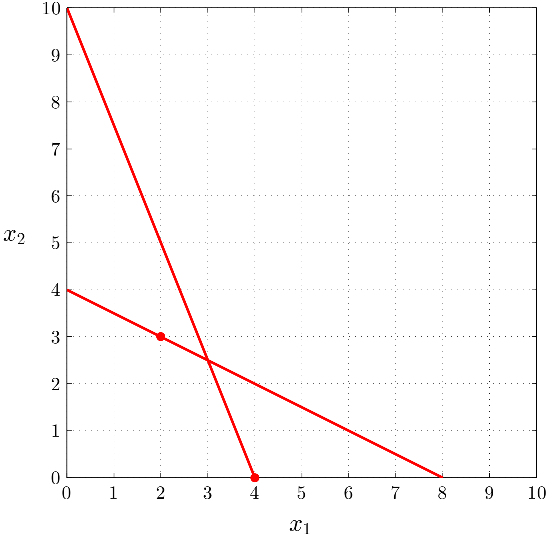


Critical Cost Efficiency Index



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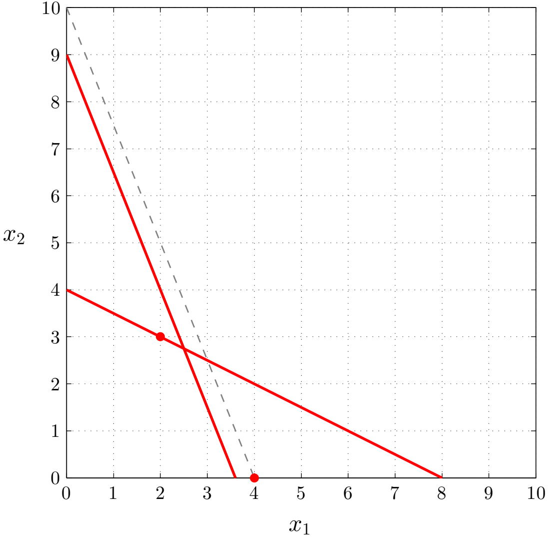

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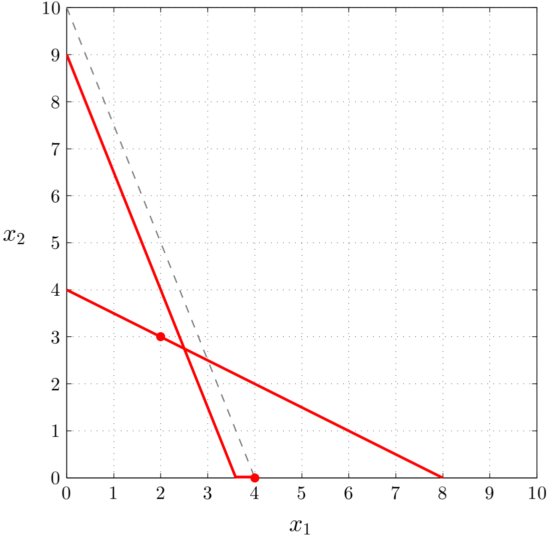

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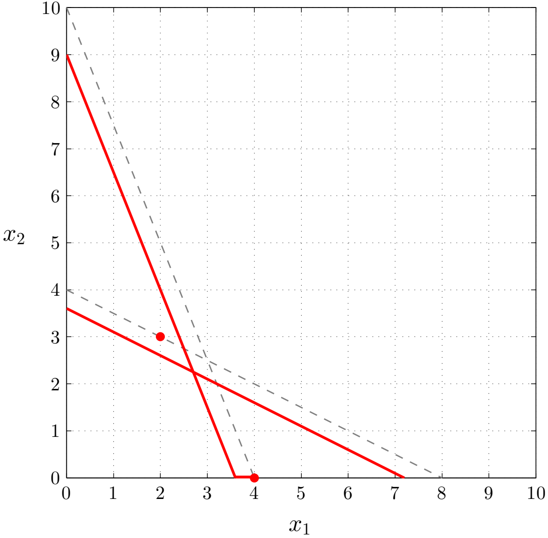

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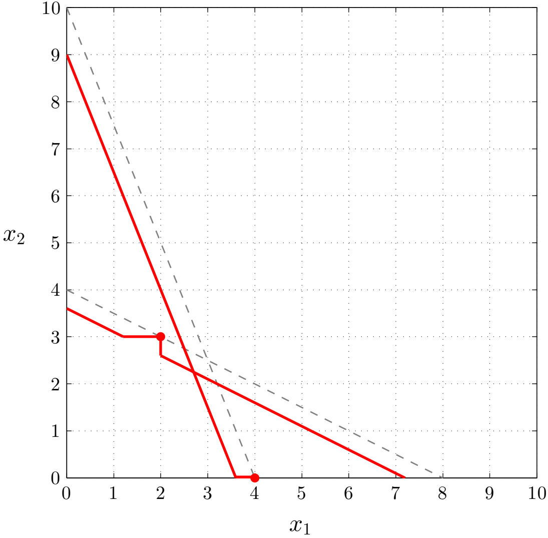

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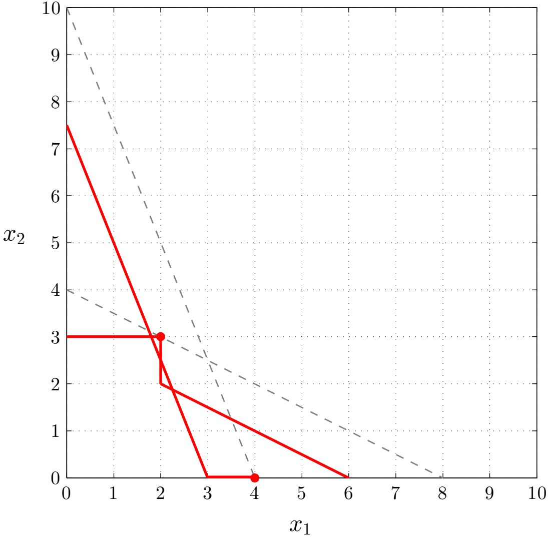

CCEI for Utility Maximization

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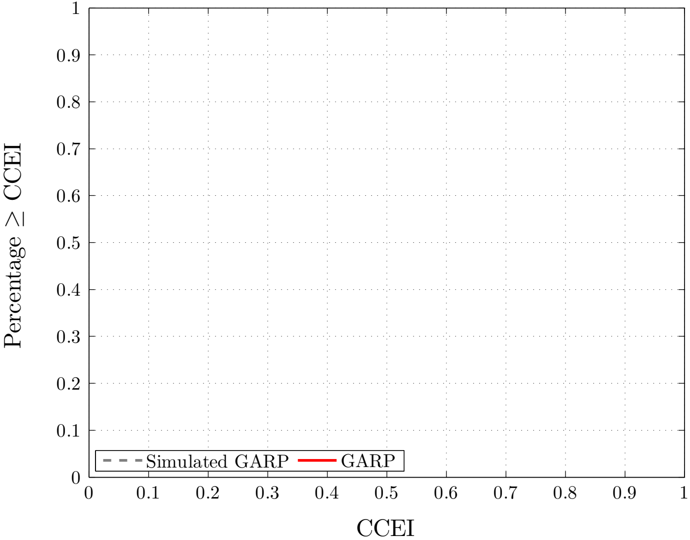

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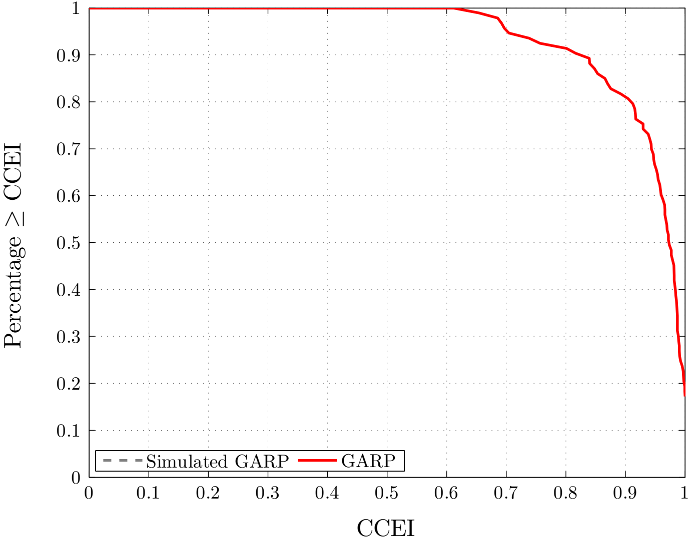

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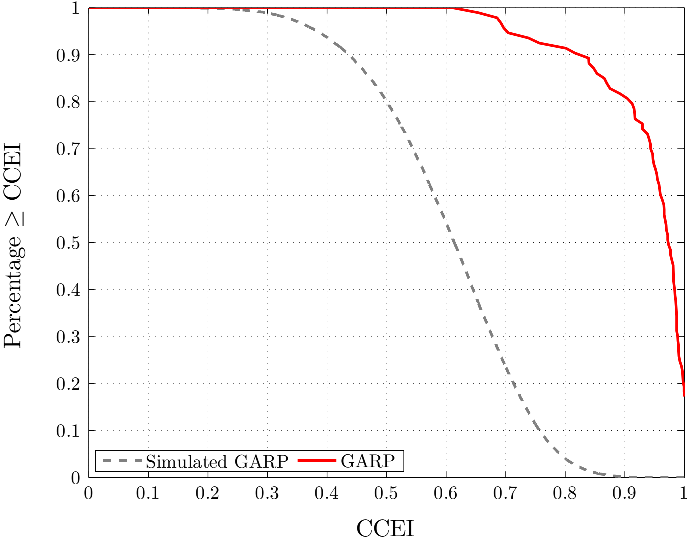

Afriat's Theorem generalized, part II

So far, we only require utility functions $U : \mathbb{R}^\ell \to \mathbb{R}$ to be strictly increasing. Formally, this means that $U(x) > U(y)$ whenever $x > y$, where $x > y$ means that $x \neq y$ and $x_i \geqslant y_i$ for all i.

But we may wish to have U increasing with some other order.

In the context of contingent consumption, it is natural to ask:

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is U strictly increasing with respect to first order stochastic dominance (i.e., if x first order stochastically dominates y, then $U(x) > U(y)$)?

First order stochastic dominance is a preorder $\gtrsim_{FSD}$ that partially orders elements in $\mathbb{R}^\ell_+$.

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Example: Suppose there are two states, where state 1 occurs with probability $1/3$ and state 2 with probability $2/3$.

Then $(1,2) >_{FSD} (2,1)$ and we may require $U(1,2) > U(2,1)$ even though $(1,2) \ngtr (1,2)$.

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Similar restrictions are relevant in intertemporal consumption. For example, we may require positive time preference.

Afriat's Theorem generalized, part II

Given a data set $\mathcal{O} = \{(p^t, x^t)\}_{t\in T}$, we write $\mathcal{X} = \{x^t\}_{t\in T}$.

Let $\succsim$ be a preorder on $\mathbb{R}^\ell_+$. For example, $\succsim = \succsim_{FSD}$.

For x^t, $x^s \in \mathcal{X}$, x^t is directly revealed preferred to x^s if there is y such that $p^t \cdot x^t \geqslant p^t \cdot y$ and $y \succsim x^t$. [Notation: $x^t \succeq^* x^s$.]

If either $p^t \cdot x^t > p^t \cdot y$ and $y \succ x^t$, we say that x^t is directly strictly revealed preferred to x^s. [Notation: $x^t \ggcurly^* x^s$.]

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Definition: $\mathcal{O} = \{(p^t, x^t)\}_{t\in T}$ obeys GARP (with respect to $\succsim$) if whenever there are observations (p^{t_i}, x^{t_i}) (for $i = 1, 2, ..., n$) satisfying

$$x^{t_1} \succeq^* x^{t_2},\ x^{t_2} \succeq^* x^{t_3},\ \ldots,\ x^{t_{n-1}} \succeq^* x^{t_n}, \text{ and } x^{t_n} \succeq^* x^{t_1},$$

then we cannot replace $\succeq^*$ with $\ggcurly^*$ anywhere in this sequence.

Afriat's Theorem generalized, part II

Examples of GARP violations:

Suppose there are just two states and $\pi_1 < \pi_2$.

1. $p^t = (3, 2)$ and $x^t = (2, 1)$.

Notice that $p^t \cdot x^t > p^t \cdot (1, 2)$ and $(1, 2) >_{FSD} (2, 1) = x^t$. Therefore,

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2. Suppose $p^t = (2, 1)$, $x^t = (3, 4)$ and $p^s = (1, 5/4)$, $x^s = (8, 0)$.

Then $p^s \cdot x^t = 8 = p^s \cdot x^s$, so $x^s \succeq^* x^t$. And $p^t \cdot x^s > p^t \cdot x^t$. So data *can* be rationalized with a strictly increasing utility function.

But $x^t \ggcurly^* x^s$ because $p^t \cdot x^t > p^t \cdot (0, 8)$ and $(0, 8) >_{FSD} (8, 0) = x^s$, so we obtain a violation of GARP, with respect to $\gtrsim_{FSD}$.

Afriat's Theorem generalized, part II

Definition: $\mathcal{O} = \{(p^t, x^t)\}_{t \in T}$ obeys GARP (with respect to $\gtrsim$) if whenever there are observations (p^{t_i}, x^{t_i}) (for $i = 1, 2, ..., n$) satisfying

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Theorem: (Nishmura, Ok, Quah) *A data set $\mathcal{O} = \{(p^t, x^t)\}_{t \in T}$ is rationalizable by a continuous utility function that is strictly increasing with $\gtrsim$ if and only if it obeys GARP (with respect to $\gtrsim$).*

Afriat's Theorem generalized, part II

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It is possible to extend the result to the case where budget sets are nonlinear as well, so this result generalizes the Forges-Minelli Theorem.

Afriat's Theorem generalized, part II

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If $\mathcal{O} = \{(p^t, x^t)\}_{t \in T}$ fails GARP (with respect to $\succsim$) then we can calculate the CCEI.

This is the largest $\alpha \in [0, 1]$ such that there is a function $U : \mathbb{R}^\ell_+ \to \mathbb{R}$, which is continuous, strictly increasing with $\succsim$, and has the following property: $U(x^t) \geqslant U(x)$ for all $x \in \mathcal{B}^t(\alpha)$, where

$$\mathcal{B}^t(\alpha) = \{x \in \mathbb{R}^\ell_+ : x \leqslant x^t\} \cup \{x \in \mathbb{R}^\ell_+ : p^t \cdot x \leqslant \alpha\, p^t \cdot x^t\}$$

(in other words, U rationalizes the data set $\mathcal{O}(\alpha) = \{(\mathcal{B}^t(\alpha), x^t)\}_{t \in T}$).

Afriat's Theorem generalized, part II

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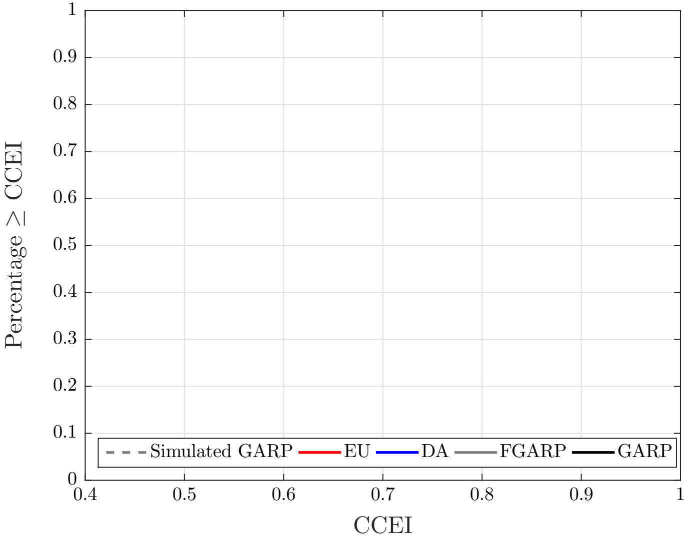
We can test for GARP with respect to $\succsim_{FSD}$ on the data collected by Choi et al. (2007), just as we did for regular GARP, and also work out the distribution of CCEIs.

CCEI for Utility Maximization



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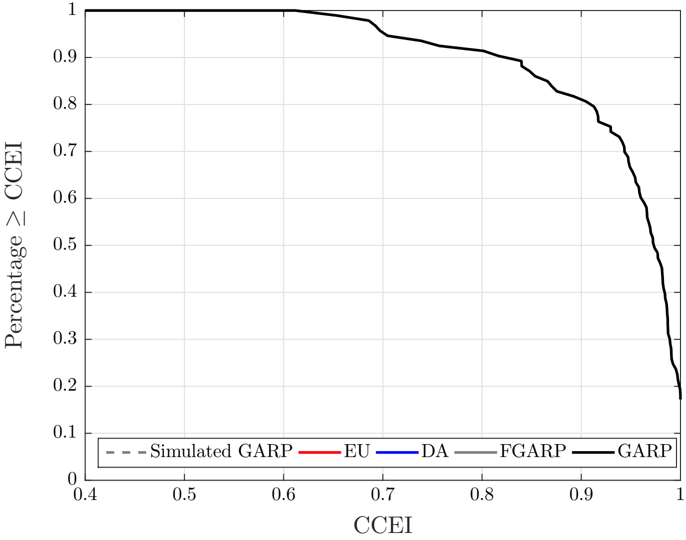

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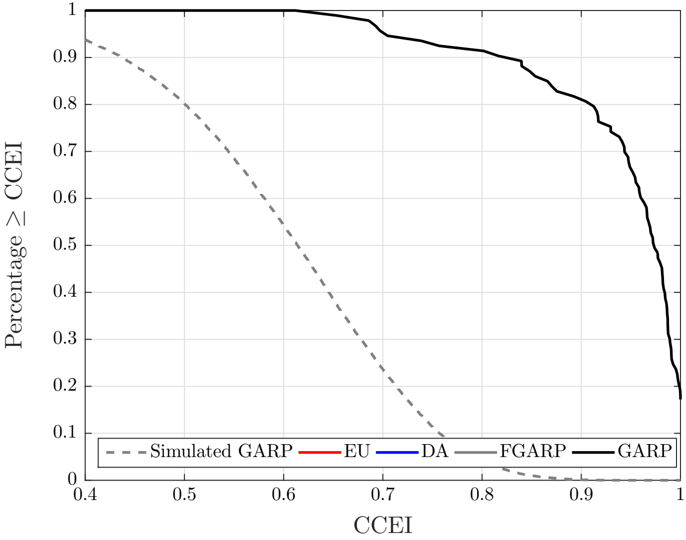

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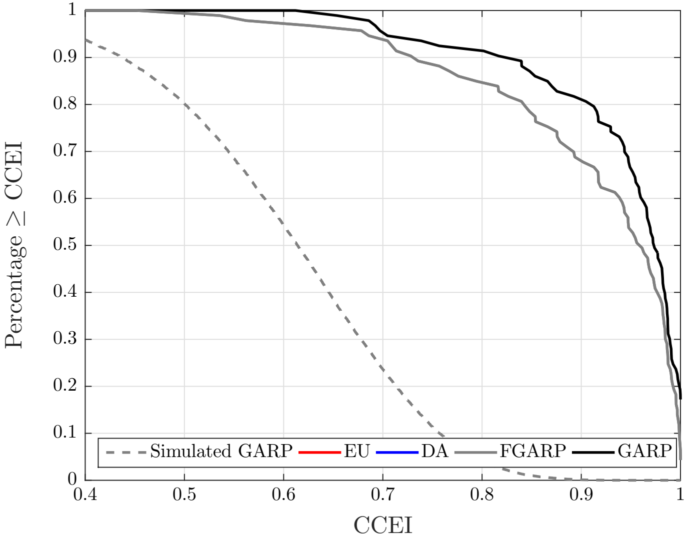

Readings

The treatment of Afriat's Theorem in these slides follow

J. K.-H. Quah (2012): "A revealed preference test for weakly separable preferences," *Economics Series Working Papers* 601, University of Oxford, Department of Economics.

The CCEI calculations on the data collected by Choi et al. (2007) are drawn from

Polisson, M., J. K.-H. Quah, and L. Renou (2015): "Revealed preferences over risk and uncertainty," *IFS Working Papers*, W15/25, Institute for Fiscal Studies, U.K.