Part I: Comparative Statics and Supermodular Games

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One-dimensional comparative statics

Let $X \subseteq \mathbb{R}$ and $f,\ g : X \to \mathbb{R}$.

We are interested in comparing $\text{argmax}_{x \in X} f(x)$ with $\text{argmax}_{x \in X} g(x)$.

Standard approach:

Assume X is a compact interval and f and g are quasi-concave functions.

Let x^* be the unique maximizer of f. Then $f'(x^*) = 0$.
Show that $g'(x^*) \geq 0$. Then optimum has shifted to the right.

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This approach makes various assumptions, most notably the quasi-concavity of f and g. This is not always the most natural assumption; for example:

let x be output, P the inverse demand function, and c the marginal cost of producing good.
The profit function $\Pi(x) = xP(x) - cx$ is not naturally concave.

One-dimensional comparative statics

Assume that f and g are continuous functions and their domain X is compact. Then $\text{argmax}_{x \in X} f(x)$ and $\text{argmax}_{x \in X} g(x)$ are nonempty. But these sets need not be singletons or intervals.

First question: how do we compare sets?

Definition: Let S' and S'' be subsets of $\mathbb{R}$. S'' dominates S' in the strong set order ($S'' \geq S'$) if for any for x'' in S'' and x' in S', we have $\max\{x'', x'\}$ in S'' and $\min\{x'', x'\}$ in S'.

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Example: $\{3, 5, 6, 7\} \not\geq \{1, 4, 6\}$ but $\{3, 4, 5, 6, 7\} \geq \{1, 3, 4, 5, 6\}$.

Note: if $S'' = \{x''\}$ and $S' = \{x'\}$, then $x'' \geq x'$.

When S'' and/or S' are non-singleton,
largest element in S'' is *larger than* the largest element in S';
smallest element in S'' is *larger than* the smallest element in S'.

One-dimensional comparative statics

Definition: g dominates f by the single crossing property ($g \succeq_{SC} f$) if for all x'' and x' such that $x'' > x'$, the following holds:

$$f(x'') - f(x') \geq (>)\, 0 \implies g(x'') - g(x') \geq (>)\, 0. \qquad (1)$$

Let S be a partially ordered set.
A family of real-valued functions $\{f(\cdot, s)\}_{s \in S}$ is an SCP family if the functions are ordered by the single crossing property (SCP), i.e., whenever $s'' > s'$, we have $f(\cdot, s'') \succeq_{SC} f(\cdot, s')$.

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Motivation for the term 'single crossing': for any $x'' > x'$, the function $\Delta : S \to \mathbb{R}$ defined by

$$\Delta(s) = f(x'', s) - f(x', s)$$

crosses the horizontal axis at most once.

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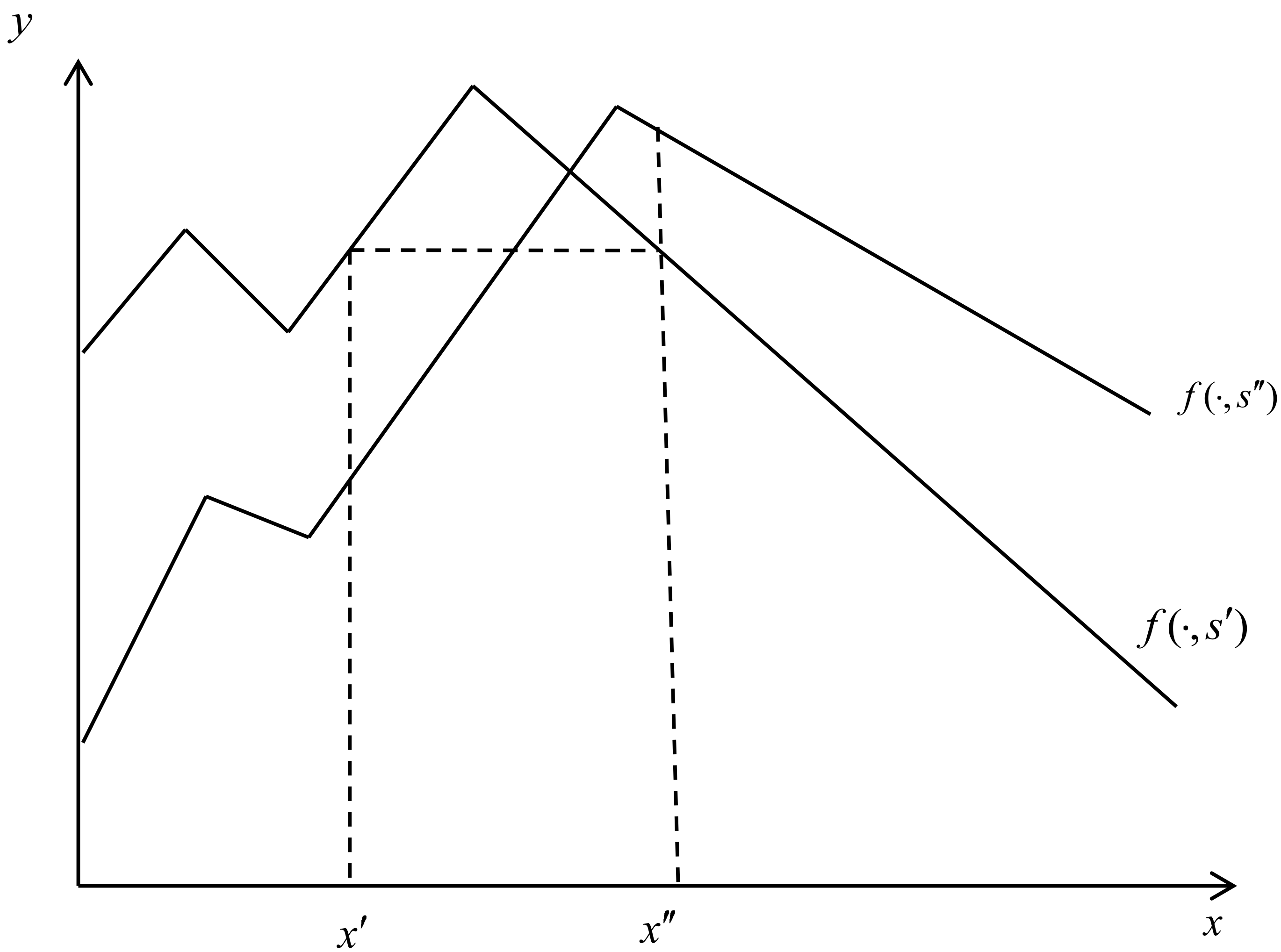
Single crossing is an ordinal property: if $g \succeq_{SC} f$, then $\phi \circ g \succeq_{SC} \psi \circ f$, where ϕ and ψ are increasing functions mapping $\mathbb{R}$ to $\mathbb{R}$.

One-dimensional comparative statics

In this case, $f(\cdot, s'') \succeq_{SC} f(\cdot, s')$



One-dimensional comparative statics

Definition: g dominates f by the increasing differences ($g \succeq_{IN} f$) if for all x'' and x' such that $x'' > x'$, the following holds:

$$g(x'') - g(x') \geq f(x'') - f(x') \tag{4}$$

Clearly, $g \succeq_{IN} f$ implies $g \succeq_{SC} f$.

Similarly, a family $\{f(\cdot, s)\}_{s \in S}$ satisfies increasing differences if the functions are ordered by increasing differences, i.e., whenever $s'' > s'$, we have $f(\cdot, s'') \succeq_{IN} f(\cdot, s')$.

One-dimensional comparative statics

Let S be an open subset of $\mathbb{R}^l$ and X an open interval. Then a sufficient (and necessary) condition for the family $\{f(\cdot, s)\}_{s \in S}$ to obey increasing differences is that

$$\frac{\partial^2 f}{\partial x \partial s_i}(x, s) \geq 0$$

at every point (x, s) and for all i.

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A sufficient condition for the family $\{f(\cdot, s)\}_{s \in S}$ to obey SCP is that there is a family $\{\bar{f}(\cdot, s)\}_{s \in S}$, where $\bar{f}(\cdot, s)$ is an increasing transformation of $f(\cdot, s)$ such that

$$\frac{\partial^2 \bar{f}}{\partial x \partial s_i}(x, s) \geq 0$$

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One-dimensional comparative statics

Theorem 1: (Milgrom-Shannon) Suppose $X \subseteq \mathbb{R}$ and $f,\ g : X \to \mathbb{R}$; then $\text{argmax}_{x \in Y} g(x) \geq \text{argmax}_{x \in Y} f(x)$ for any $Y \subseteq X$ if and only if $g \succeq_{SC} f$.

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Proof: Assume $g \succeq_{SC} f$, $x'' \in$ argmax$_{x \in Y} f(x)$, and $x' \in$ argmax$_{x \in Y} g(x)$. We have to show that $\max\{x', x''\} \in$ argmax$_{x \in Y} g(x)$ and $\min\{x', x''\} \in$ argmax$_{x \in Y} f(x)$. We need only consider the case where $x'' > x'$.

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Since x'' is in $\text{argmax}_{x \in Y} f(x)$, we have $f(x'') \geq f(x')$ and since $g \succeq_{SC} f$, we also have $g(x'') \geq g(x')$; thus x'' is in $\text{argmax}_{x \in Y} g(x)$. Furthermore, $f(x'') = f(x')$ so that x' is in $\text{argmax}_{x \in Y} f(x)$. If not, $f(x'') > f(x')$ which implies (by the fact that $g \succeq_{SC} f$) that $g(x'') > g(x')$, contradicting the assumption that g is maximized at x'.

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Necessity: clear that if SCP property is violated at x' and x'', then

$$\text{argmax}_{x \in \{x', x''\}} g(x) \not\geq \text{argmax}_{x \in \{x'. x''\}} f(x).$$

One-dimensional comparative statics

Application: Recall $\Pi(x, -c) = xP(x) - cx$. Then $\{\Pi(\cdot, -c)\}_{-c \in \mathbb{R}_-}$ obey increasing differences, since

$$\frac{\partial^2 \Pi}{\partial x \partial c} = -1.$$

By Theorem 1, $\text{argmax}_{x \in X} \Pi(x, -c)$ is increasing in $-c$.

For example, $\text{argmax}_{x \in X} \Pi(x, -3) \geq \text{argmax}_{x \in X} \Pi(x, -5)$.

In other words, the profit-maximizing output decreases as the marginal cost of output increases.

One-dimensional comparative statics

Application: Bertrand Oligopoly with differentiated products, with

$$\begin{aligned}\Pi_a(p_a, p_{-a}) &= (p_a - c_a)D_a(p_a, p_{-a}) \\ \ln \Pi_a(p_a, p_{-a}) &= \ln(p_a - c_a) + \ln D_a(p_a, p_{-a})\end{aligned}$$

So $\{\Pi_a(\cdot, p_{-a})\}_{-a \in -A}$ has SCP if $\{\ln \Pi_a(\cdot, p_{-a})\}_{-a \in -A}$ has increasing differences. The latter condition requires

$$\frac{\partial^2}{\partial p_a \partial p_{-a}} [\ln \Pi_a] \geq 0,$$

which is equivalent to

$$\frac{\partial^2}{\partial p_a \partial p_{-a}} [\ln D_a] \geq 0.$$

One-dimensional comparative statics

The condition

$$\frac{\partial^2}{\partial p_a \partial p_{-a}} [\ln D_a] \geq 0$$

is in turn equivalent to

$$\frac{\partial}{\partial p_{-a}} \left[-\frac{p_a}{D_a} \frac{\partial D_a}{\partial p_a} \right] \leq 0;$$

i.e., firm a's own-price elasticity of demand,

$$\epsilon_a(p_a, p_{-a}) = -\frac{p_a}{D_a} \frac{\partial D_a}{\partial p_a} \text{ decreases with } p_{-a}.$$

If this assumption holds, $\text{argmax}_{p_a \in P} \Pi_a(p_a, p_{-a})$ increases with p_{-a}.

In other words, firm a's optimal price is *increasing* in the price charged by other firms. Firms' strategies are complements.

Supermodular Games

The Bertrand game is an example of a supermodular game.

A supermodular game is one where the best response of each agent is increasing with the strategies of the other agents.

In the next few slides we take a brief look at the properties of these games.

We do not assume that the payoff functions of the agents in the game are quasiconcave. Therefore, the best response map need not be a convex-valued correspondence. For this reason, the ‘standard’ proof of equilibrium existence via Kakutani’s fixed point theorem cannot be applied.

Instead, we appeal to the monotonicity of the best response map and use another fixed point theorem - Tarski’s.

Supermodular Games

Let $X = \Pi_{i=1}^{N} X_i$, where each X_i is a compact interval.

Theorem 2: (Tarski) Suppose $\phi : X \to X$ is an increasing function. Then the set of fixed points $\{x \in X : \phi(x) = x\}$ is nonempty. In fact, $x^{**} = \sup\{x \in X : x \leq \phi(x)\}$ is a fixed point and is the largest fixed point, i.e., for any other fixed point x^*, we have $x^* \leq x^{**}$.

Note: ϕ need not be continuous.

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Theorem 3: Suppose $\phi(\cdot, t) : X \to X$ is increasing in (x, t). Then the largest fixed point of $\phi(\cdot, t)$ is increasing in t.

Supermodular Games

Theorem 3: Suppose $\phi(\cdot, t) : X \to X$ is increasing in (x, t). Then the largest fixed point of $f(\cdot, t)$ is increasing in t.

Proof: By Theorem 2, the largest fixed point

$$\bar{x}(t) = \sup\{x \in X : x \leq \phi(x, t)\}.$$

Suppose $t'' > t'$; since $\phi(x, t'') \geq \phi(x, t')$ for all x,

$$\{x \in X : x \leq \phi(x, t')\} \subseteq \{x \in X : x \leq \phi(x, t'')\}.$$

Therefore,

$$\sup\{x \in X : x \leq \phi(x, t')\} \leq \sup\{x \in X : x \leq \phi(x, t'')\}.$$

By Theorem 2 again, $\bar{x}(t') \leq \bar{x}(t'')$. QED

Supermodular Games

Bertrand Oligopoly: assume the set of firms is A; the typical firm a chooses its price from the compact interval P to maximize $\Pi_a(p_a, p_{-a}) = (p_a - c_a)D_a(p_a, p_{-a})$.

Recall: if own-price elasticity is decreasing in p_{-a} then the family $\{D_a(\cdot, p_{-a})\}_{p_{-a}}$ forms an SCP family.

Define $B_a(p_{-a}) = \text{argmax}_{p_a \in P} \Pi_a(p_a, p_{-a})$.
This is the set of a's best responses to the pricing of other firms. It is nonempty if Π_a is continuous.

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Define $\bar{B}_a(p_{-a}) = \max\left[\text{argmax}_{p_a \in P} \Pi_a(p_a, p_{-a})\right]$;
This is the *largest* best response to the pricing of other firms. This is an increasing function of p_{-a} (by Theorem 1).

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Define $\bar{P} = P \times P \times ... \times P$ and the map $\bar{B} : \bar{P} \to \bar{P}$ by

$$\bar{B}(p) = \left(\bar{B}_a(p_{-a})\right)_{a \in A}.$$

A fixed point of this map is a NE of the game.

Supermodular Games

By Theorem 1, $\bar{B}$ is an increasing function.

By Tarski's Fixed Point Theorem, a fixed point exists.

Specifically, $p^* = \sup\{p \in \bar{P} : p \leq \bar{B}(p)\}$ is a fixed point of the map $\bar{B}$ and thus a NE. In fact, this is the *largest* NE. Why?

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Let $\hat{p} = (p_a)_{a \in A}$ be any other NE. Then $\hat{p}_a \in B_a(\hat{p}_{-a})$. So

$$\hat{p}_a \leq \bar{B}_a(\hat{p}_{-a}),$$

and we obtain $\hat{p} \leq \bar{B}(\hat{p})$. Thus, $\hat{p} \in \{p \in \bar{P} : p \leq \bar{B}(p)\}$, which implies $\hat{p} \leq p^* \equiv \sup\{p \in \bar{P} : p \leq \bar{B}(p)\}$.

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In this game, best response function is monotonic but not necessarily continuous. Existence guaranteed by appealing to Tarski's fixed point theorem, rather than Kakutani's. The game has a *largest* NE. We can do comparative statics exercises on the largest NE...

Supermodular Games

What happens to the largest NE when firm $\tilde{a}$ experiences an increase in marginal cost from $c_{\tilde{a}}$ to $c'_{\tilde{a}}$? Recall

$$\begin{aligned} \Pi_{\tilde{a}}(p_{\tilde{a}}, p_{-\tilde{a}}, c_{\tilde{a}}) &= (p_{\tilde{a}} - c_{\tilde{a}}) D_{\tilde{a}}(p_{\tilde{a}}, p_{-\tilde{a}}) \\ \ln \Pi_{\tilde{a}}(p_{\tilde{a}}, p_{-\tilde{a}}, c_{\tilde{a}}) &= \ln(p_{\tilde{a}} - c_{\tilde{a}}) + \ln D_{\tilde{a}}(p_{\tilde{a}}, p_{-\tilde{a}}) \end{aligned}$$

Observe that

$$\frac{\partial}{\partial p_{\tilde{a}} \partial c_{\tilde{a}}} [\ln \Pi_{\tilde{a}}] > 0.$$

By Theorem 1, firm a's best response increase with $c_{\tilde{a}}$ (for fixed p_{-a}). Formally,

$$B_{\tilde{a}}(p_{-\tilde{a}}, c'_{\tilde{a}}) \geq B_{\tilde{a}}(p_{-\tilde{a}}, c_{\tilde{a}}).$$

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$$B_{\tilde{a}}(p_{-\tilde{a}}, c'_{\tilde{a}}) \geq B_{\tilde{a}}(p_{-\tilde{a}}, c_{\tilde{a}}).$$

This implies that $\bar{B}(p, c'_{\tilde{a}}) \geq \bar{B}(p, c_{\tilde{a}})$. So largest fixed point of $\bar{B}(\cdot, c'_{\tilde{a}})$ is larger than the largest fixed point of $\bar{B}(\cdot, c_{\tilde{a}})$ (by Theorem 3). In other words, if firm $\tilde{a}$'s marginal cost increases from $c_{\tilde{a}}$ to $c'_{\tilde{a}}$, the largest NE increases: *every firm increases its price*.

Multidimensional comparative statics

Definitions:

We endow $\mathbb{R}^l$ with the product order: $x' \geq x$ if $x'_i \geq x_i$ for all i.

An element y in $\mathbb{R}^l$ is an upper bound of $S \subset \mathbb{R}^l$ if $y \geq x$ for all x in S.

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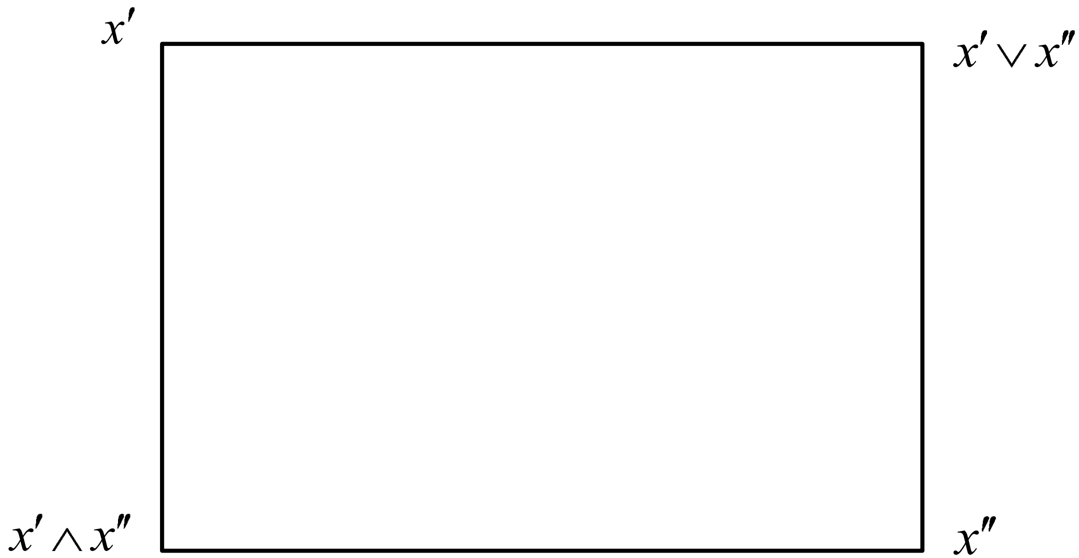
Note:

(1) $\sup S$ is *unique* if it exists. (2) $\sup S$ need not be in S.

Multidimensional comparative statics

We denote the supremum of x' and x'' by $x' \vee x''$ and their infimum by $x' \wedge x''$. Easy to check that $x' \vee x'' = (\max\{x'_1, x''_1, \max\{x'_2, x''_2\}, ..., \max\{x'_l, x''_l\})$ and $x' \wedge x'' = (\min\{x'_1, x''_1, \min\{x'_2, x''_2\}, ..., \min\{x'_l, x''_l\})$.



A set S is a sublattice of $\mathbb{R}^l$ if for any x' and x'' in S, $x' \vee x''$ and $x' \wedge x''$ are both in S.

Multidimensional comparative statics

More definitions:

Let S' and S'' be subsets of $\mathbb{R}^l$. S'' dominates S' in the strong set order ($S'' \geq S'$) if for any for x'' in S'' and x' in S', we have $x' \vee x''$ in S'' and $x' \wedge x''$ in S'.

Multidimensional comparative statics

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Let S' and S'' be subsets of $\mathbb{R}^l$. S'' dominates S' in the strong set order ($S'' \geq S'$) if for any for x'' in S'' and x' in S', we have $x' \vee x''$ in S'' and $x' \wedge x''$ in S'.

Note:

(1) If $S'' = \{x''\}$ and $S' = \{x'\}$, then $S'' \geq S'$ if and only if $x'' \geq x'$.

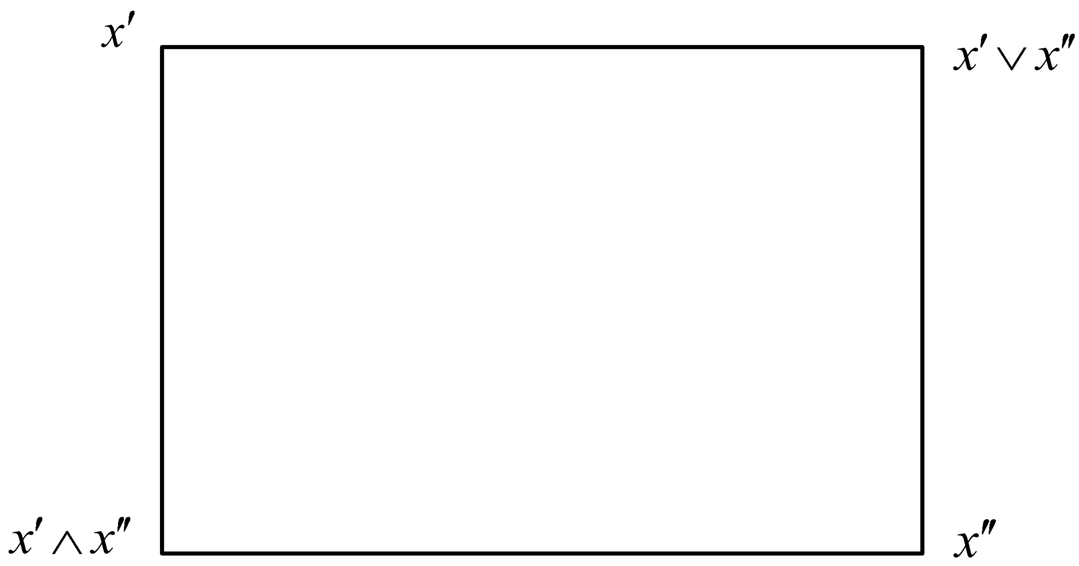
(2) Suppose S' and S'' both contain their suprema; then $S'' \geq S'$ implies that $\sup S'' \geq \sup S'$.

Proof of (2): Let x'' and x' be suprema of S'' and S' respectively. Then $x'' \vee x'$ is in S''. Since x'' is $\sup S''$, we have $x'' \geq x'' \vee x'$, which implies $s'' \geq x'$. QED

Multidimensional comparative statics

Let X be a sublattice of $\mathbb{R}^l$ and $f : X \to \mathbb{R}$. The function f is supermodular (SPM) if

$$f(x' \vee x'') - f(x'') \geq f(x') - f(x' \wedge x'').$$



Multidimensional comparative statics

Let X be a sublattice of $\mathbb{R}^l$ and $f : X \to \mathbb{R}$. The function f is supermodular (SPM) if

$$f(x' \vee x'') - f(x'') \geq f(x') - f(x' \wedge x'').$$

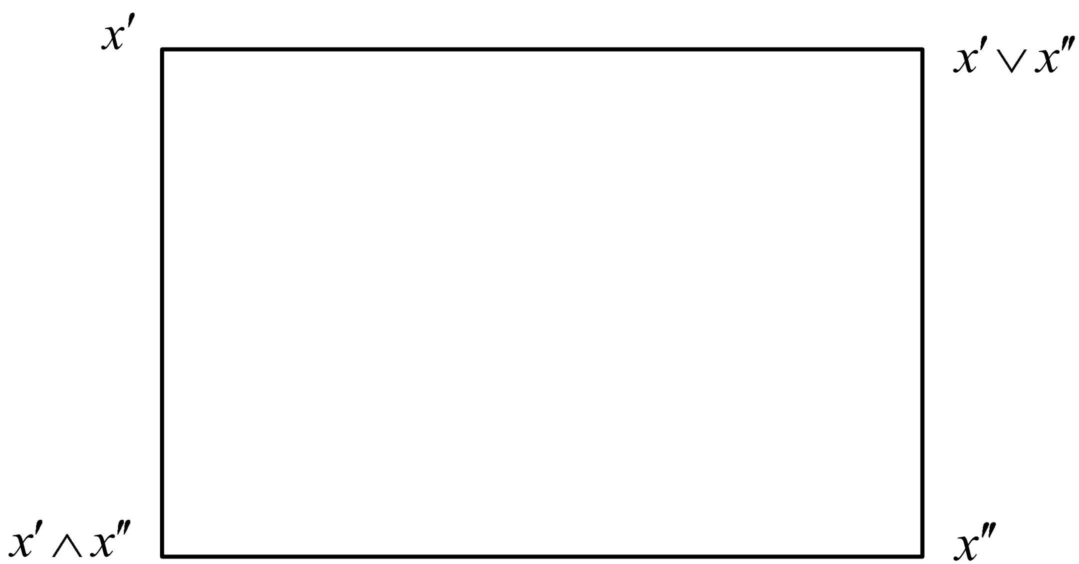
The function f is quasisupermodular (QSM) if

$$f(x') - f(x' \wedge x'') \geq \ (>)0 \Longrightarrow f(x' \vee x'') - f(x'') \geq \ (>)0.$$

Multidimensional comparative statics

Theorem 4: Let X be a sublattice of $\mathbb{R}^l$ and suppose that $f : X \to \mathbb{R}$ is a QSM function. Then $\text{argmax}_{x \in X} f(x)$ is a sublattice.

Proof: Suppose x' and x'' are both in $\text{argmax}_{x \in X} f(x)$. Since X is a sublattice, $x' \vee x''$ and $x' \wedge x''$ are both in is in X. Note that $f(x') \geq f(x' \wedge x'')$ since $x' \in \text{argmax}_{x \in X} f(x)$. The QSM property guarantees that $f(x' \vee x'') \geq f(x'')$, which implies that $x' \vee x'' \in \text{argmax}_{x \in X} f(x)$. Etc. QED



Multidimensional comparative statics

Recall:

g dominates f by the single crossing property ($g \succeq_{SC} f$) if for all x'' and x' such that $x'' > x'$, the following holds:

$$f(x'') - f(x') \geq (>)\, 0 \implies g(x'') - g(x') \geq (>)\, 0. \tag{5}$$

and g dominates f by the increasing differences ($g \succeq_{IN} f$) if for all x'' and x' such that $x'' > x'$, the following holds:

$$g(x'') - g(x') \geq f(x'') - f(x') \tag{6}$$

These definitions still make sense when x is multidimensional.

Multidimensional comparative statics

[1] Sufficient condition for $f : X \to \mathbb{R}$ to be supermodular:

$$\frac{\partial^2 f}{\partial x_i \partial x_j}(x) \geq 0 \text{ at every point } x \in X\text{, for } i \neq j$$

(assuming X is open in $\mathbb{R}^l$).

A family $\{f(\cdot, s)\}_{s \in S}$ satisfies increasing differences if the functions are ordered by increasing differences, i.e., whenever $s'' > s'$, we have $f(\cdot, s'') \succeq_{IN} f(\cdot, s')$.

[2] Sufficient condition for $\{f(\cdot, s)\}_{s \in S}$ to satisfy increasing differences:

$$\frac{\partial^2 f}{\partial x_i \partial s_j}(x, s) \geq 0$$

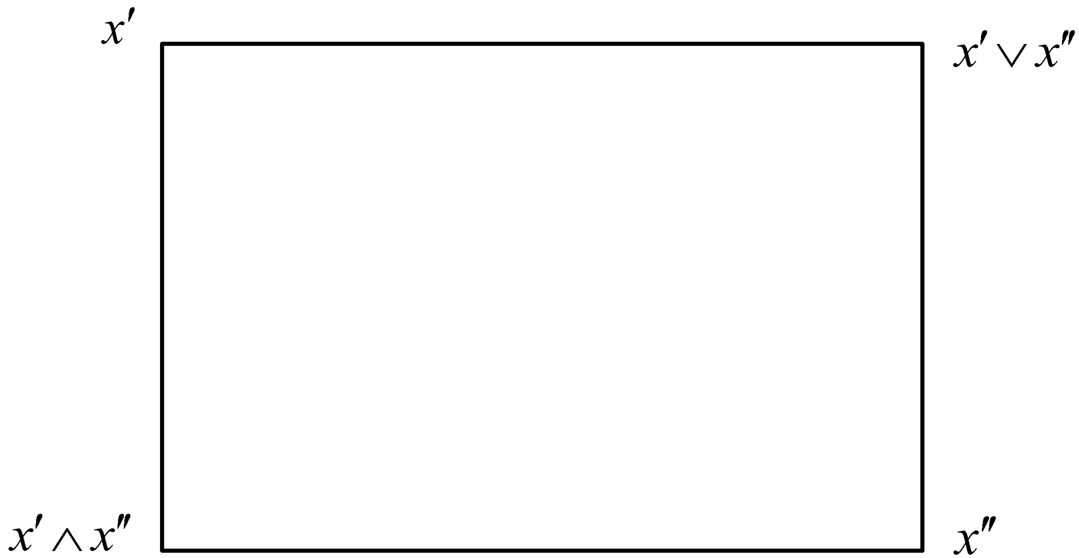
at every point $(x, s) \in X \times S$ and for all i and j (assuming X and S are both open subsets of Euclidean spaces).

The next result is the multidimensional version of Theorem 1.

Multidimensional comparative statics

Theorem 5: (Milgrom and Shannon) Suppose $X \subseteq \mathbb{R}$ is a sublattice and $f,\ g : X \to \mathbb{R}$ two QSM functions.
Then $\text{argmax}_{x \in Y} g(x) \geq \text{argmax}_{x \in Y} f(x)$ for every sublattice Y contained in X if and only if $g \succeq_{SC} f$.

Proof: Let $x' \in \text{argmax}_{x \in Y} f(x)$ and $x'' \in \text{argmax}_{x \in Y} g(x)$. Then $f(x') \geq f(x' \wedge x'')$. By QSM of f, $f(x' \vee x'') \geq f(x'')$. Since $g \succeq_{SC} f$, we obtain $g(x' \vee x'') \geq g(x'')$, which implies that $x' \vee x''$ is in $\text{argmax}_{x \in Y} g(x)$. Etc. QED



Multidimensional comparative statics

Application:

Let x denote the vector of inputs (drawn from $X = \mathbb{R}^l_+$), p the vector of input prices, and V the revenue function mapping input vector x to revenue (in $\mathbb{R}$). The firm's profit is

$$\Pi(x;p) = V(x) - p \cdot x.$$

Note that

$$\frac{\partial^2 \Pi}{\partial x_i \partial p_j}(x;p) = -1.$$

So there is decreasing differences.

If V is supermodular, then Π is supermodular.

Conclusion:
By Theorem 5, $\text{argmax}_{x \in X} \Pi(x;p') \geq \text{argmax}_{x \in X} \Pi(x;p'')$ if $p' < p''$.